$L-$-Series and Their 2-adic and 3-adic Valuations at s=1 Attached to CM Elliptic Curves [*][†]

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Abstract

$L-$series of two classical families of elliptic curves with complex multiplication are studied, formulae for their special values at $s = 1$, bound of the values, and criterion of reaching the bound are given. Let $E_D: \ y^2 = x^3 - Dx$ be elliptic curves over the Gaussian field $K = \mathbb{Q}(\sqrt{-1})$, with $D = \pi_1 \cdots \pi_n$ or $D = \pi_1^2 \cdots \pi_r^2 \pi_{r+1} \cdots \pi_n$ (where $\pi_1, \ \cdots, \ \pi_n$ are distinct primes in K. A formula for special values of Hecke $L-$series of such curves expressed by Weierstrass $\wp-$function are given; a lower bound of 2-adic valuations of these values of Hecke $L-$series as well as a criterion for reaching these bounds are obtained; moreover the first part of the conjecture of Birch and Swinnerton-Dyer is hence verified to hold for some of these curves. Let $E_{D^2}: \ y^2 = x^3 - 2^4 3^3 D^2$ and $E_{D^3}: \ y^2 = x^3 + 2^4 D^3$ be elliptic curves over the field $\mathbb{Q}(\sqrt{-3})$ (with $D = \pi_1 \cdots \pi_n$, where $\pi_1, \ \cdots, \ \pi_n$ are distinct primes of $\mathbb{Q}(\sqrt{-3})$), similar results as above for $3 - adic$ valuation are obtained. These results develop some results for more special case and for $2 - adic$ valuation.

I. Introduction and Statement of Main Results

Consider the two classical families of elliptic curves:

$$E_1: \qquad y^2 = x^3 - Dx;$$

$$E_2: \qquad y^2 = x^3 + D',$$

here $D, \ D' \in \mathbb{Z}$ are rational integers (but in the following we will generalize to the cases that $D, \ D'$ are certain quadratic algebraic numbers). These elliptic curves have been studied broadly for a long time, having relations with many problems of number theory. For example, the curve

[*] 2000 Mathematics Subject Classification: Primary 11G05; Secondary 11G40, 14H52, 14G10, 14K22

[†] Key words and phrases: elliptic curve, complex multiplication, L-series, L-function, valuation, BSD conjecture

E_1 correlates intimately to the problem of congruent numbers when D is a square in $\mathbb{Z}$(see [Tun]). These two families of elliptic curves E have complex multiplication by $\sqrt{-1}$ and $\sqrt{-3}$ respectively, their complex $L-$series (or $L-$function) $L(E,\ s)$ could be identified with the $L-$series attached to certain Hecke characters (i.e. Grössencharacter) of the fields $\mathbb{Q}\ (\sqrt{-1})$ and $\mathbb{Q}\ (\sqrt{-3})$ respectively. The "conjecture of Birch and Swinnerton-Dyer" (or " B-SD conjecture", for brevity) asserts that the value of the $L-$series $L(E,\ s)$ at $s=1$ of an elliptic curve $E,\ \ L(E,\ 1),$ is very important for the arithmetic study of the elliptic curve. A considerable of (numerical) evidences for the B-SD conjecture have been held up since it was published, most of them were from the above two families of elliptic curves as could be seen in the original paper of Birch and Swinnerton-Dyer [B-SD] , and papers of Razar [Razar ??] and Stephens [Ste] . In particular, if D is a perfect square of integer, then the B-SD conjecture predicts that $L(E_1,\ 1)$ could be divided by a power of 2 (up to a multiple of an appropriate period of E_1) which depends on the number of distinct prime factors of D. For certain kinds of $D',$ the B-SD conjecture has similar prediction for the curve E_2 (see e.g. [Razar] , [Tun] , [Ste]).

In 1997, C. Zhao studied the problem of divisibility of $L(E_1,\ 1)$ by powers of 2 under the new assumption $D=D_0^2$ with D_0 an Gaussian integer in the Gaussian field $\mathbb{Q}(\sqrt{-1})$. In fact, he studied the $2-adic$ valuation of the value at $s=1$ of the $L-$series of the elliptic curve $E_1:\quad y^2=x^3-D_0^2x,\ D_0\in\mathbb{Z}[\sqrt{-1}]$ (Actually, the value of the $L-$series should be divided by an appropriate period Ω of the elliptic curve first; this normally will not be mentioned again in the following as a default fact); he gave the rigorous lower bound for the $2-adic$ valuation as well as a criterion of reaching this bound, and hence obtained nice results about congruent numbers and showed the first part of the B-SD conjecture is true for some elliptic curves $E_{D_0}^2$ over Gaussian field.

In 1968, N. Stephens studied a case of the elliptic curves $E_2,$ i.e. $E:\quad y^2=x^3-2^43^3D_1^2$ with $D_1\in\mathbb{Z}$([Ste]). He proved that if $D_1>2$ is a cube-free rational integer, then $\psi(D_1)=3^{1/2}D_1^{1/3}L(E,\ 1)/\Omega$ is a rational integer (where Ω is the period mentioned above, a constant expressed by Weierstrass $\wp-$functions), and

3 divides $\psi(D_1)=3^{1/2}D_1^{1/3}L(E,\ 1)/\Omega\qquad$ (when $9|D_1$).

In the present paper, we will study the two classical families of elliptic curves E_1 and E_2 further on the base fields $K_1=\mathbb{Q}(\sqrt{-1})$ and $K_1=\mathbb{Q}(\sqrt{-3})$ respectively, giving formulae of values of their $L-$series at $s=1,$ lower bound for their $2-adic$ and $3-adic$ valuations , criteria for reaching the bounds, and verify B-SD conjecture in some cases.

Over the Gaussian field $\mathbb{Q}(\sqrt{-1}),$ we will first study the elliptic curves $E_D:\ y^2=x^3-Dx$

with $D=\pi_1\cdots\pi_n$ and $D=\pi_1^2\cdots\pi_r^2\pi_{r+1}\cdots\pi_n$ (where $\pi_1, \cdots, \pi_n$ are distinct Gaussian prime integers in $\mathbb{Z}(\sqrt{-1})$ (when $r=n$, the second case turns to be the case studied by Zhao [Zhao]). We will give a formula for the special values at $s=1$ of the Hecke $L-$Series of E_D (expressed via Weierstrass $\wp-$function), lower bounds for the $2-adic$ valuation of the values, criterion of reaching the bounds, and show that the B-SD conjecture about the relation of the rank of the Mordell-Weil group and the analytic rank of the $L-$series is true for some elliptic curves E_D, by using our criterion and results of Coates and Wiles .

Then over the quadratic field $\mathbb{Q}(\sqrt{-3})$, we consider the two kinds of elliptic curves E_{D^2} : $y^2=x^3-2^4 3^3 D^2$, and E_{D^3} : $y^2=x^3+2^4D^3$ with $D=\pi_1\cdots\pi_n$ where $\pi_1, \cdots, \pi_n$ are distinct prime integers in $K_2=\mathbb{Q}(\sqrt{-3})$. Similar results as above (but for $3-adic$ valuation) will be given. These results develop the results about the estimation of $2-adic$ valuation for special value of $L-$series of E_D with $D=D_0^2$ square in Gaussian field in [Zhao] .

(A.1) Now let $K=\mathbb{Q}(\sqrt{-1})$ be the Gaussian field, $O_K=\mathbb{Z}(\sqrt{-1})$ the ring of its integers (Gaussian integers), and put $I=\sqrt{-1}$. Consider the elliptic curve

$$E_D: \quad y^2=x^3-Dx \qquad D=\pi_1\cdots\pi_n,$$

where $\pi_1, \cdots, \pi_n$ are distinct prime (Gaussian) integers in $O_K=\mathbb{Z}(\sqrt{-1})$ and $\pi_k\equiv 1(mod\ 4)$ $(k=1, \cdots, n)$. So $D\equiv 1(mod\ 4)$.

Denote the set $S=\{\pi_1, \cdots, \pi_n\}$. For any subset T of the set $\{1, \cdots, n\}$, define

$$D_T=\prod_{k\in T}\pi_k, \qquad \widehat{D}_T=\prod_{k\notin T}\pi_k=D/D_T\ ,$$

and put $D_\emptyset=1$ when $T=\emptyset$ (emptyset). Let

$$L_S(\overline{\psi}_{D_T},\ s)$$

denote the Hecke $L-$series of ψ_{D_T} (omitting all the Euler factors corresponding to primes in S), where ψ_{D_T} is the Hecke character (Grössencharacter) of the field K corresponding to the elliptic curve E_{D_T} : $y^2=x^3-D_Tx$. For the special value $L_S(\overline{\psi}_{D_T},\ 1)$ of the above $L-$series at $s=1$, we have the following formula expressed as a finite sum of Weierstrass $\wp-$function $\wp(z)$.

Theorem 1.1 For any factor D_T of $D=\pi_1\cdots\pi_n\in\mathbb{Q}(\sqrt{-1})$ as above , let ψ_{D_T} be the Hecke character of the Gaussian field $\mathbb{Q}(\sqrt{-1}$ corresponding to the elliptic curve E_{D_T} : $y^2=x^3-D_Tx$. Then we have

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S\left(\overline{\psi}_{D_T},\ 1\right)=\frac{I}{2}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\frac{1}{\wp\left(\frac{c\omega}{D}\right)-I}+\frac{1}{4}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \qquad (1.1)$$

where $\theta = 2+2I$, $\left(\frac{\alpha}{\beta}\right)_4$ denote the (generalized) quartic residue symbol, $\mathcal{C}$ is any complete reduced remainder-system of O_K modulo D, $L_\omega = \omega O_K$ is the period lattice of the elliptic curve $E_1 :\ y^2 = x^3 - x$,

$$\omega = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} = 2.6220575\cdots ,$$

$\wp(z)$ is the Weierstrass $\wp-$function associated to the lattidce L_ω.

Let $\overline{\mathbb{Q}_2}$ be the algebraic closure of the $2-adic$ (complete) field $\mathbb{Q}_2$, $v = v_2$ is the normalized $2-adic$ exponential valuation of $\overline{\mathbb{Q}_2}$ (i.e. $v_2(2) = 1$). Fix an isomorphic embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_2}$, where $\overline{\mathbb{Q}}$ is the algebraic closue of the rational field $\mathbb{Q}$. For any algebraic number α, let $v_2(\alpha)$ denote the $2-adic$ valuation of α. For $D = \pi_1\cdots\pi_n$ as above, put

$$S^*(D) = \frac{I}{2}\sum_{c\in\mathcal{C}} \frac{1}{\wp(\frac{c\ \omega}{D}) - I} \sum_T \left(\frac{c}{D_T}\right)_4 . \qquad (1.2)$$

For any Gaussian integers α, β which are relatively prime, put $\left(\frac{\alpha}{\beta}\right)_4^2 = \left(\frac{\alpha}{\beta}\right)_2$, and define $\left[\frac{\alpha}{\beta}\right]_2 = (1 - \left(\frac{\alpha}{\beta}\right)_2)/2$, then $\left[\frac{\alpha\gamma}{\beta}\right]_2 = \left[\frac{\alpha}{\beta}\right]_2 + \left[\frac{\gamma}{\beta}\right]_2$
(regard $[-]_2$ as a $\mathbb{F}_2-$value function).

For $D = \pi_1\cdots\pi_n$, we define a $\mathbb{F}_2-$value function δ_k inductively as the following : First put

$$\varepsilon_n(D) = \begin{cases} 1, & \text{if} \quad v_2(S^*(D)) = (n-1)/2 \quad ; \\ 0, & \text{if} \quad v_2(S^*(D)) > (n-1/2 \ , \end{cases}$$

where $n = n(D)$ is the number of distinct prime factors of D. Then for Gaussian prime integer π with $\pi \equiv 1 (mod\ 4)$, define s_1 as a $\mathbb{F}_2-$valued function as follows:

$$s_1(\pi) = \begin{cases} 1, & \text{if} \quad v_2(\pi - 1) = 2 \quad ; \\ 0, & \text{if} \quad v_2(\pi - 1) > 2 \quad . \end{cases}$$

Finally define the $\mathbb{F}_2-$value function δ_k $(k = 1, 2, \cdots)$ as follows:

$$\delta_1(\pi) = s_1(\pi) + \varepsilon_1(\pi);$$

and for $D = \pi_1\cdots\pi_n$ $(n \geq 2)$, define

$$\delta_n(D) = \delta_n(\pi_1, \cdots, \pi_n) = \varepsilon_n(D) + \sum_{\emptyset\neq T \subsetneqq \{1,\cdots,n\}} \left(\prod_{k\notin T} \left[\frac{D_T}{\pi_k}\right]_2\right) \delta_t(D_T) ,$$

where the sum " $\sum$" is taken over the nonempty subsets T of $\{1, \cdots, n\}$, and $t = \sharp T$ is the cardinal of T.

Theorem 1.2. Let $D = \pi_1 \cdots \pi_2$, where $\pi_k \equiv 1(mod\ 4)$ are distinct Gaussian prime integers $(k = 1, \cdots, n)$. Then for the $2-adic$ valuation of the values of the $L-$series we have

$$v_2\left(L(\overline{\psi}_D, 1)/\omega\right) \geq \frac{n-1}{2},$$

and the equality holds if and only if $\delta_n(D) = 1$.

Theorem 1.3. Let $D = \pm p_1 \cdots p_m \equiv 1(mod\ 4)$, where $p_k \not\equiv 5(mod\ 8)$ are distinct positive rational prime numbers $(k = 1, \cdots, m)$. If $\delta_n(D) = 1$, then the first part of B-SD conjecture is true for the elliptic curve $E_D:\ y^2 = x^3 - Dx$, that is

$$rank(E_D(\mathbb{Q})) = ord_{s=1}(L(E_D/\mathbb{Q}, s)) = 0. \qquad (1.3)$$

(where n is the number of distinct prime factors of D).

(A.2) Now consider the elliptic curves $E_D:\ y^2 = x^3 - Dx$ for $D = \pi_1^2 \cdots \pi_r^2 \pi_{r+1} \cdots \pi_n$, where $\pi_k \equiv 1(mod\ 4)$ are distinct prime Gaussian integers $(k = 1,\ \cdots,\ n)$. Similarly to tht above, let $S = \{\pi_1,\ \cdots,\ \pi_n\}$. Write any subset T of $\{1,\ \cdots,\ n\}$ as $T = T_1 \cup T_2$, where $T_1 = T \cap \{1,\ \cdots,\ r\}$, $T_2 = T \cap \{r+1,\ \cdots,\ n\}$. And define

$$D_T = D_{T_1} D_{T_2} \quad , \quad D_{T_1} = \prod_{k\ \in T_1} \pi_k^2 \quad , \quad D_{T_2} = \prod_{k \in T_2} \pi_k \ .$$

$$\widehat{D}_T = \frac{D}{D_T} = \widehat{D}_{T_1} \widehat{D}_{T_2} \qquad D_1 = \pi_1^2 \cdots \pi_r^2\ , \quad D_2 = \pi_{r+1} \cdots \pi_n\ , \quad D = D_1 D_2.$$

$$\widehat{D}_{T_1} = \frac{D_1}{D_{T_1}}\ , \quad \widehat{D}_{T_2} = \frac{D_2}{D_{T_2}}.$$

When $T = \emptyset$(emptyset)(or $T_1 = \emptyset$, or $T_2 = \emptyset$), define $D_T = 1$ (or $D_{T_1} = 1$, or $D_{T_2} = 1$ respectively). (If $r = n$, then $D_1 = D$, $D_2 = 1$; and if $r = 0$, then $D_1 = 1$, $D_2 = D$).
For above $D = \pi_1^2 \cdots \pi_r^2 \pi_{r+1} \cdots \pi_n$, set S and T, denote

$$\Delta_1 = \pi_1 \cdots \pi_r \qquad \Delta_2 = \pi_{r+1} \cdots \pi_n \qquad \Delta = \pi_1 \cdots \pi_r \pi_{r+1} \cdots \pi_n = \Delta_1 \Delta_2\ ,$$

$$\Delta_T = \prod_{k \in T_1} \pi_k \cdot \prod_{k \in T_2} \pi_k = \Delta_{T_1} \Delta_{T_2}\ , \quad \Delta_{T_1} = \prod_{k \in T_1} \pi_k\ , \quad \Delta_{T_2} = \prod_{k \in T_2} \pi_k.$$

$$\widehat{\Delta}_T = \frac{\Delta}{\Delta_T}\ , \quad \widehat{\Delta}_{T_1} = \frac{D_1}{\Delta_{T_1}}\ , \quad \widehat{\Delta}_{T_2} = \frac{D_2}{\Delta_{T_2}}.$$

Define $\Delta_\emptyset = 1$.

Let $L_S(\overline{\psi}_{D_T},\ s)$ denote Hecke $L-$series of ψ_{D_T} (omitting all Euler factors corresponding to primes in S), where ψ_{D_T} is the Hecke character of the Gaussian field $K = \mathbb{Q}(\sqrt{-1})$ corresponding

to the elliptic curve $\quad E_{D_T}: \ y^2 = x^3 - D_T x \quad$. By the definition we know

$$L_S(\overline{\psi}_{D_T}, \ s) = \begin{cases} L(\overline{\psi}_D, \ s) & \text{if} \quad D_T = D \quad ; \\ L(\overline{\psi}_{D_T}, \ s) \prod_{\pi_k | \widehat{D}_T} \left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{\overline{\pi}_k}{(\pi_k \overline{\pi}_k)^s}\right) & \quad \text{otherwise.} \end{cases}$$

Theorem 1.4. For any factor $D_T = D_{T_1} DT_2$ of $D = \pi_1^2 \cdots \pi_r^2 \pi_{r+1} \cdots \pi_n \in \mathbb{Q}(\sqrt{-1}$ as above , let ψ_{D_T} be the Hecke character of the Gaussian field corresponding to the elliptic curve $E_{D_T}: \ \ y^2 = x^3 - D_T x$. Then we have

$$\frac{\Delta}{\omega} \overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, \ 1) = \frac{I}{2} \sum_{c \in \mathcal{C}} \left(\frac{c}{D_T}\right)_4 \frac{1}{\wp\left(\frac{c\ \omega}{\Delta}\right) - I} + \frac{1}{4} \sum_{c \in \mathcal{C}} \left(\frac{c}{D_T}\right)_4, \qquad (1.)$$

where $\Delta =$???????????????????????????????????

$\mathcal{C}$ is a complete reduced residue system of O_K modulo Δ; $\theta = 2 + 2I, \ \ L_\omega = \omega O_K, \ \omega,$ and $\wp(Z)$ are as in Theorem 1.1.

Theorem 1.5. Let $D = \pi_1^2 \cdots \pi_r^2 \pi_{r+1} \cdots \pi_n, \ \ n, \ r$ are positive integers $(1 \leq r \leq n)$ (If $r = n$ then $D = \pi_1^2 \cdots \pi_n^2$), $\pi_k \equiv 1 (mod\ 4)$, are distinct prime Gaussian integers $(k = 1, \ \cdots, \ n)$. Then for the $2 - adic$ valuation of the values of the $L-$series we have

$$v\left(L(\overline{\psi}_D, \ 1)/\omega\right) \geq \frac{n}{2} - 1. \qquad (1.5)$$

(B) Now we consider the elliptic curves $y^2 = x^3 - D'$ over the number field $K = \mathbb{Q}(\sqrt{-3})$ with complex multiplication by $\sqrt{-3}$. Let $\tau = (-1 + \sqrt{-3})/2 = exp(2\pi I/3)$ be a primitive cubic root of unity, $O_K = \mathbb{Z}[\tau]$ be the ring of integers of K. We will study elliptic curve

$$E_{D^2}: \quad y^2 = x^3 - 2^4 3^3 D^2 ,$$

where $D = \pi_1 \cdots \pi_n, \ \ \pi_k \equiv 1 (mod\ 6)$ are distinct prime elements of O_K ($k = 1, \cdots, n0$.

Let $S = \{\pi_1, \cdots, \pi_n\}$. For any subset T of $\{1, \cdots, n\}$, define

$$D_T = \prod_{k \in T} \pi_k \qquad , \quad \widehat{D}_T = \prod_{k \notin T} \pi_k = D/D_T ,$$

and put $D_\emptyset = 1$. Let $\psi_{D_T^2}$ be the Hecke character of of K corresponding to the elliptic curve

$$E_{D_T^2}: \ y^2 = x^3 - 2^4 3^3 D_T^2.$$

And let $L_S(\overline{\psi}_{D_T^2},\ s)$ denote the Hecke $L-$series of $\psi_{D_T^2}$ (omitting all the Euler factors corresponding to primes in S). Then $L_S(\overline{\psi}_{D_T^2},\ 1)$ could be expressed by the Weierstrass $\wp-$functions as in the following:

Theorem 1.6. For any factor D_T of $D=\pi_1\cdots\pi_n \in \mathbb{Q}(\sqrt{-3})$ as above , let $\psi_{D_T^2}$ be the Hecke character of the field $\mathbb{Q}(\sqrt{-3})$ corresponding to the elliptic curve $E_{D_T^2}:\ y^2=x^3-2^4 3^3 D_T^2$. Then we have

$$\frac{D}{\omega}\left(\frac{9}{D_T}\right)_3 L_S(\overline{\psi}_{D_T^2},1)=\frac{1}{2\sqrt{3}}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_3\frac{1}{\wp\left(\frac{c\omega}{D}\right)-1}+\frac{1}{3\sqrt{3}}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_3. \quad (1.6)$$

where $\wp(z)$ is the Weierstrass
$wp-$function associated to $L_\omega=\omega O_K$ the period lattice of the elliptic curve $E_1:\ y^2=x^3-\frac{1}{4}$, $\mathcal{C}$ is a complete residue system of O_K modulo D, $\omega=3.059908\cdots$ is a constant, $\left(\frac{a}{b}\right)_3$ is the cubic residue symbol.

Now let $\overline{\mathbb{Q}}_3$ be the algebraic closure of $\mathbb{Q}_3$, the $3-adic$completion of $\mathbb{Q}$. Let v_3 be the normalized $(3-adic)$ exponential valuation of $\overline{\mathbb{Q}}_3$, i.e. $v_3(3)=1$. Fix an (isomorphic) embedding $\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}_3}$.

Theorem 1.7. Let $D=\pi_1\cdots\ \pi_n$, where $\pi_k\equiv 1(mod\ 6)$, are distinct prime elements of the number field $K=\mathbb{Q}(\sqrt{-3})$ $(k=1,\ \cdots,\ n\ )$. Then we have

$$v_3\left(L(\overline{\psi}_{D^2},\ 1)/\omega\right)\geq\frac{n}{2}-1. \quad (1.7)$$

Theorem 1.8. For $D=\pi_1\cdots\pi_n$ as in Theorem 1.7, put

$$S^*(D)=\frac{1}{2\sqrt{3}}\sum_{c\ \in\mathcal{C}}\frac{1}{\wp(\frac{c\ \omega}{D},\ L_\omega)-1}\sum_T 2^{n-t(T)}\ \left(\frac{c}{D_T}\right)_3. \quad (1.8)$$

Then we have

$$v_3(S^*(D))\geq\frac{n-1}{2}.$$

II. 2-Valuations of $L-$series of Elliptic Curves with CM by $\sqrt{-1}$

We need the following results :

Proposition A Let E be an elliptic curve defined over the imaginary quadratic field K with complex multiplication ring O_K(integers in K). Assume its period lattice is $L=\Omega O_K$, $\Omega\in\mathbb{C}^\times$ a complex number, ϕ is the Hecke character of K corresponding to E, $\mathfrak{g}$ is an integral ideal of K , $E_{\mathfrak{g}}$

is the subgroup of E consist of $\mathfrak{g}-$divisible points. Let $\mathbf{B}$ be a set of integral ideals of K relatively prime to $\mathfrak{g}$ and

$$\{\sigma_\flat \mid \flat \in \mathbf{B}\} = Gal(K(E_\mathfrak{g})/K), \quad (\text{if } \flat \neq \flat', \text{ then } \sigma_\flat \neq \sigma_{\flat'})$$

where

$$\sigma_\flat = \left(\frac{K(E_\mathfrak{g})/K}{\flat}\right)$$

is *Artin* symbol. Put $\rho \in \Omega K^\times \subset \mathbb{C}^\times$, and $\rho\Omega^{-1}O_K = \mathfrak{g}^{-1}\mathfrak{h}$, $\mathfrak{h}$ is an integral ideal of K relatively priime to $\mathfrak{g}$. Then

$$\frac{\phi^k(\mathfrak{h})}{N(\mathfrak{h})^{k-s}} \cdot \frac{\overline{\rho}^k}{|\rho|^{2s}} \cdot L_\mathfrak{g}(\overline{\phi}^k, \quad s) = \sum_{\flat \in \mathbf{B}} H_k(\phi(\flat)\rho, \ 0, \ s, \ L)$$

($Re(s) > 1 + k/2$), where k is an positive integer, N denotes the norm map from K to $\mathbb{Q}$.

$$L_\mathfrak{g}(\overline{\phi}^k, \ s) = \prod_{\wp \dagger \mathfrak{g}} (1 - \overline{\phi}^k(\wp)N(\wp)^{-s})^{-1}, \qquad (Re(s) > 1 + k/2)$$

$$H_k(z, \ 0, \ s, \ L) = \sum{}' \frac{(\overline{z} + \overline{\alpha})^k}{|z + \alpha|^{2s}}, \qquad (Re(s) > 1 + k/2)$$

where the sum $\sum'$ is taken over $\alpha \in L = \Omega O_K$ and $\alpha \neq -z$ when $z \in L$ [Go-Sch] .

Lemma B. Let elliptic curve E, field K, Hecke character ϕ, and $\mathfrak{g}$ are as in Proposition A. If the conductor f_ϕ of ϕ divides $\mathfrak{g}$, then $K(E_\mathfrak{g})$ is the ray class field of K to the cycle (or divisor, modulo) $\mathfrak{g}$ (see [Go-Sch]).

Now we consider Theorem 1.1 and let $K = \mathbb{Q}(\sqrt{-1})$, E_D, D_T, and $L_S(\overline{\psi}_{D_T}, \ s)$ etc be as there. Then by definition (see [B-SD] , [Ire-Ro]) we have

Lemma 2.1.

$$L_S(\overline{\psi}_{D_T}, \ s) = \begin{cases} L(\overline{\psi}_{D_T}, \ s), & \text{if } \prod_{\pi_k \in S} \pi_k = D_T \quad ; \\ L(\overline{\psi}_{D_T}, \ s) \prod_{\pi_k | \widehat{D}_T} (1 - \left(\frac{D_T}{\pi_k}\right)_4 \cdot \frac{\overline{\pi}_k}{(\pi_k\overline{\pi}_k)^s}), & \text{otherwise .} \end{cases}$$

Proof of Theorem 1.1. For the elliptic curve E_{D_T} : $y^2 = x^3 - D_T x$, assume its period lattice is $L = \Omega O_K$, with $\Omega = \alpha\omega$, $\alpha \in \mathbb{C}^\times$ (Obviously $\Omega = \omega/\sqrt[4]{D_T}$). From [Bir-Ste] we know the conductor of ψ_{D_T} is (θD_T). Now, in Proposition A, let $k = 1$, $\rho = \Omega/(\theta D)$, $\mathfrak{g} = (\theta D)$, $\mathfrak{h} = O_K$, we have

$$\frac{\overline{\rho}}{|\rho|^{2s}} L_\mathfrak{g}(\overline{\psi}_{D_T}, \ s) = \sum_{\flat \in \mathbf{B}} H_1(\psi_{D_T}(\flat)\rho, 0, s, L), \quad (Re(s)) > 3/2). \qquad (2.1)$$

Since the conductor of ψ_{D_T} is θD_T, and $(\theta D_T) \mid (\theta D) = \mathfrak{g}$, so by Lemma B we know that the ray class field of K to the cycle (θD) is $K((E_{D_T})_{(\theta D)})$, in particular we have the following isomorphism via Artin map:

$$(O_K/(\theta D))^\times/\mu_4 \cong Gal(K((E_{D_T})_{(\theta D)})/K),$$

where μ_4 is the group of quartic roots of unity, and $\mu_4 \cong (O_K/\theta)^\times$. So we could take the set

$$\mathbf{B} = \{(c\theta + D) \quad | \quad c \in \mathcal{C}\}, \qquad (2.2)$$

where $\mathcal{C}$ are fixed representations of $(O_K/(D))^\times$, so we have

$$\frac{\overline{\rho}}{|\rho|^{2s}} L_{\mathfrak{g}}(\overline{\psi}_{D_T}, \ s) = \sum_{c \in \mathcal{C}} H_1(\psi_{D_T}(c\theta + D)\rho, 0, s, L), \quad (Re(s) > 3/2) \qquad (2.3)$$

Note that the analytic extension of $H_1(z, o, 1, L)$ could be obtained by Eisenstein E^*- (see [Zhao] or [We]), i.e., $H_1(z, 0, 1, L) = E^*_{0,1}(z, L) = E^*_1(z, L)$, So by (2.3) we have

$$\frac{\theta D}{\Omega} L_{(\theta D)}(\overline{\psi}_{D_T}, \quad 1) = \sum_{c \in \mathcal{C}} E^*_1(\psi_{D_T}(c\theta + D))\frac{\Omega}{\theta D}, \quad \Omega O_K). \qquad (2.4)$$

Since $D \equiv 1(mod\ 4)$, so $c\theta + D \equiv 1(mod\ \theta)$ for any $c \in \mathcal{C}$. Thus by the definition of ψ_{D_T} and quartic reciprocity we have

$$\psi_{D_T}(c\theta + D) = \overline{\left(\frac{D_T}{c\theta + D}\right)_4}(c\theta + D) = \overline{\left(\frac{c\theta + D}{D_T}\right)_4}(c\theta + D) = \overline{\left(\frac{c\theta}{D_T}\right)_4}(c\theta + D).$$

Then by (2.4) and the fact $L_{(\theta D)}(\overline{\psi}_{D_T}, \quad 1) = L_S(\overline{\psi}_{D_T}, \quad 1)$, we have

$$\frac{\theta D}{\alpha\omega} L_S(\overline{\psi}_{D_T}, \quad 1) = \sum_{c \in \mathcal{C}} E^*_1\left(\left(\frac{c\omega}{D} + \frac{\omega}{\theta}\right)\alpha\overline{\left(\frac{c\theta}{D_T}\right)_4}, \quad \alpha\omega O_K\right). \qquad (2.5)$$

Put $\lambda = \alpha\overline{\left(\frac{c\theta}{D_T}\right)_4}$, by

$$E^*_1(\lambda z, \quad \lambda L) = \lambda^{-1} E^*_1(z, \quad L),$$

we have

$$E^*_1((\frac{c\omega}{D} + \frac{\omega}{\theta})\alpha\overline{\left(\frac{c\theta}{D_T}\right)_4}, \quad \alpha\overline{\left(\frac{c\theta}{D_T}\right)_4}\omega O_K) = \frac{1}{\alpha}\left(\frac{c\theta}{D_T}\right)_4 E^*_1(\frac{c\omega}{D} + \frac{\omega}{\theta}, \quad \omega O_K)$$

So by (2.5) we have

$$\frac{\theta D}{\omega} L_S(\overline{\psi}_{D_T}, \quad 1) = \left(\frac{\theta}{D_T}\right)_4 \sum_{c \in \mathcal{C}} \left(\frac{c}{D_T}\right)_4 E^*_1(\frac{c\omega}{D} + \frac{\omega}{\theta}, \quad \omega O_K), \qquad (2.6)$$

For the period lattice $L_\omega = \omega O_K$ mentioned above, denote the corresponding Weierstrass $\wp-$function by $\wp(z, L_\omega)$, denote the corresponding Weierstrass $Zeta-$function by $\zeta(z, L_\omega)$, then we have $\wp'(z)^2 = 4\wp(z)^3 - 4\wp(z)$. So by results in [Go-Sch] we have

$$E_1^*(\frac{c\omega}{D} + \frac{\omega}{\theta}, \omega O_K) = \zeta(\frac{c\omega}{D}, L_\omega) + \zeta(\frac{\omega}{\theta}, L_\omega) + \frac{1}{2}\frac{\wp'(\frac{c\omega}{D}) - (2-2I)}{\wp(\frac{c\omega}{D}) - I} - \frac{\pi}{\omega}\overline{(\frac{c}{D} + \frac{1}{\theta})}. \qquad (2.7)$$

The representation system $\mathcal{C}$ of $(O_K/(D))^\times$ may be so chosen that $-c \in \mathcal{C}$ whenever $c \in \mathcal{C}$. Then $\left(\frac{-c}{D_T}\right)_4 = \left(\frac{c}{D_T}\right)_4$. Since $\zeta(z, L_\omega)$ and $\wp'(z, L_\omega)$ are odd functions, and $\wp(z, L_\omega)$ is even, so by (2.6) we have

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1) = \frac{1}{\theta}\left\{\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \zeta(\frac{c\,\omega}{D}, L_\omega) - \frac{\pi}{\omega}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{\overline{c}}{\overline{D}} + \right.$$

$$\left.\frac{1}{2}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{\wp'(\frac{c\omega}{D})}{\wp(\frac{c\,\omega}{D}) - I} - (1-I)\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{1}{\wp(\frac{c\,\omega}{D}) - I}\right\}$$

$$+\frac{1}{\theta}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \left(\zeta\left(\frac{\omega}{\theta}, L_\omega\right) - \frac{\pi}{\omega\overline{\theta}}\right)$$

$$= -\frac{1-I}{\theta}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{1}{\wp(\frac{c\,\omega}{D}) - I} + \frac{1}{\theta}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \left(\zeta\left(\frac{\omega}{\theta}, L_\omega\right) - \frac{\pi}{\omega\overline{\theta}}\right)$$

That is

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1) = \frac{I}{2}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{1}{\wp(\frac{c\,\omega}{D}) - I} +$$

$$\frac{1}{\theta}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \left(\zeta\left(\frac{\omega}{\theta}, L_\omega\right) - \frac{\pi}{\omega\overline{\theta}}\right). \qquad (2.8)$$

By [Zhao] we know $\zeta(\frac{\omega}{\theta}, L_\omega) - \frac{\pi}{\omega\overline{\theta}} = \frac{\theta}{4}$, so

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1) = \frac{I}{2}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{1}{\wp(\frac{c\,\omega}{D}) - I} + \frac{1}{4}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4.$$

This proves Theorem 1.1.

Lemma 2.2 . $\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 = \begin{cases} \sharp\mathcal{C}, & \text{if} \quad T = \emptyset\ ; \\ 0, & \text{if} \quad T \neq \emptyset\ . \end{cases}$

Proof . By the definition of quartic residue symbol the lemma could be easily verified.

Lemma 2.3 . Let $D = \pi_1 \cdots \pi_n$ where $\pi_k \equiv 1(mod\ 4)$ are distinct Gaussian prime ($k = 1, \cdots, n$). Let c be any Gaussian integer relatively prime to D. Then

(1) $\sum\limits_{T}\left(\frac{c}{D_T}\right)_4 = \mu(1+I)^t$ or 0, where $\mu \in \{\pm 1, \pm I\}$, t is a integer with $n \leq t \leq 2n$.

(2) $\sum\limits_T \left(\frac{c}{D_T}\right)_4 = 0$ if and only if $\left(\frac{c}{\pi_k}\right)_4 = -1$ (for some $k \in \{1, \cdots, n\}$).

(3) Suppose that $\left(\frac{c}{\pi_k}\right)_4 \neq -1$ for arbitrary $k \in \{1, \cdots, n\}$),then

$$\sum_T \left(\frac{c}{D_T}\right)_4 = \mu(1+I)^{n+s}, \qquad \text{where } \mu \text{ is as in (1) above,}$$

$$s = \sharp\{\pi_k \ : \ \pi_k | D \quad \text{and} \quad \left(\frac{c}{\pi_k}\right)_4 = 1, \quad k = 1, \cdots, n\}.$$

In particular we know

$$\sum_T \left(\frac{c}{D_T}\right)_4 = 2^n \quad \text{if and only if} \quad \left(\frac{c}{\pi_1}\right)_4 = \cdots = \left(\frac{c}{\pi_n}\right)_4 = 1 \quad ;$$

$$\sum_T \left(\frac{c}{D_T}\right)_4 = \mu(1+I)^n \quad \text{if and only if} \quad \left(\frac{c}{\pi_k}\right)_4 \in \{I, -I\}, \quad k = 1, \cdots, n \ ,$$

where the sum $\sum\limits_T$ is taken over all subsets T of $\{1, \cdots, n\}$.

Proof . In fact we have $\sum\limits_T \left(\frac{c}{D_T}\right)_4 = \left(1 + \left(\frac{c}{\pi_1}\right)_4\right) \cdots \left(1 + \left(\frac{c}{\pi_n}\right)_4\right)$, from which the results could be deduced.

Lemma 2.4 . $v\,(S * (D)) \geq (n-1)/2$.

Proof . By results of [Zhao] or [B-SD], we know

$$v_2\left(\wp(\frac{c\omega}{D}) - I\right) = \frac{3}{4}$$

(for any Gaussian integer c relatively prime to D). And by Lemma 2.3 we have

$$v_2\left(\sum_T \left(\frac{c}{D_T}\right)_4\right) = v_2(\mu(1+I)^t) = \frac{t}{2} \geq \frac{n}{2}$$

(Here we regard $v_2(0)$ as ∞). Thus by properties of valuation and our choice of $\mathcal{C}$ with the property $c, -c \in \mathcal{C}$, we have

$$v_2(S^*(D)) \geq -\frac{3}{4} + \frac{n}{2}.$$

since $\pi_k \equiv 1(mod\ 4) \quad (k = 1, \cdots, n)$, so

$$N(D_T) \equiv N(D) \equiv (mod\ 8), \quad \left(\frac{I}{D_T}\right)_4 = I^{(N(D_T)-1)/4} = \pm 1 \ .$$

Also we have

$$\sharp(O_K/(D))^\times = \sharp\mathcal{C} = \prod_{k=1}^{n} (N(\pi_k) - 1) \equiv 0(mod\ 8) \ ,$$

so we could choose $\mathcal{C}$ properly such that $\pm c, \pm Ic \in \mathcal{C}$ (when $c \in \mathcal{C}$). Put

$$V = \{c \in \mathcal{C} \ : \ c \equiv 1(mod\ \theta)\}, \qquad V' = V \cup IV,$$

then $\mathcal{C} = V' \cup (-V')$. Since $IO_K = O_K$, so $IL_\omega = I(\omega O_K) = \omega O_K = L_\omega$. Thus by the definition of Weierstrass $\wp-$function, we could obtain

$$\wp(Iz,\ IL_\omega) = \frac{1}{(Iz)^2} + \sum_{\alpha \in IL_\omega} \left(\frac{1}{(Iz-\alpha)^2} - \frac{1}{\alpha^2} \right)$$

$$= \frac{1}{(Iz)^2} + \sum_{\alpha' \in L_\omega} \left(\frac{1}{(Iz-I\alpha')^2} - \frac{1}{(I\alpha')^2} \right)$$

$$= \frac{1}{I^2} \left(\frac{1}{z^2} + \sum_{\alpha' \in L_\omega} \left(\frac{1}{(z-\alpha')^2} - \frac{1}{\alpha'^2} \right) \right) = -\wp(z,\ L_\omega) \ ,$$

that is

$$\wp(Iz,\ L_\omega) = -\wp(z,\ L_\omega).$$

In particular, assume $z = \frac{c\omega}{D}$, then we have

$$\wp\left(\frac{Ic\ \omega}{D},\ L_\omega\right) = -\wp\left(\frac{c\ \omega}{D}, \quad L_\omega\right).$$

$$S^*(D) = \frac{I}{2} \sum_{c \in \mathcal{C}} \frac{1}{\wp(\frac{c\ \omega}{D}) - I} \sum_T \left(\frac{c}{D_T}\right)_4 = I \sum_{c \in V'} \frac{1}{\wp(\frac{c\ \omega}{D}) - I} \sum_T \left(\frac{c}{D_T}\right)_4$$

$$= I \sum_{c \in V} \left[\frac{1}{\wp(\frac{c\ \omega}{D}) - I} \sum_T \left(\frac{c}{D_T}\right)_4 + \frac{1}{\wp(\frac{Ic\ \omega}{D}) - I} \sum_T \left(\frac{Ic}{D_T}\right)_4 \right]$$

$$= I \sum_{c \in V} \left[\sum_T \frac{1}{\wp(\frac{c\ \omega}{D}) - I} \left(\frac{c}{D_T}\right)_4 + \sum_T \frac{1}{-\wp(\frac{c\ \omega}{D}) - I} \left(\frac{I}{D_T}\right)_4 \left(\frac{c}{D_T}\right)_4 \right]$$

$$= I \sum_{c \in V} \left[\sum_T \left(\frac{1}{\wp(\frac{c\ \omega}{D}) - I} - \left(\frac{I}{D_T}\right)_4 \frac{1}{\wp(\frac{c\ \omega}{D}) + I} \right) \left(\frac{c}{D_T}\right)_4 \right].$$

Note that $v_2(\wp((c\ \omega)/D) - I) = 3/4$, so we know $v_2(\wp((c\ \omega)/D) + I) = 3/4$. Note also

$$\frac{1}{\wp(\frac{c\ \omega}{D}) - I} + \frac{1}{\wp(\frac{c\ \omega}{D}) + I} = \frac{2\wp(\frac{c\ \omega}{D})}{(\wp(\frac{c\ \omega}{D}))^2 + 1} \ ,$$

$$\frac{1}{\wp(\frac{c\ \omega}{D}) - I} - \frac{1}{\wp(\frac{c\ \omega}{D}) + I} = \frac{2I}{(\wp(\frac{c\ \omega}{D}))^2 + 1} \ ,$$

$$\left(\frac{I}{D_T}\right)_4 = \pm 1 \ ,$$

so

$$S^*(D) = I \sum_{c \in V} \frac{2B}{(\wp(\frac{c\ \omega}{D}))^2 + 1} \sum_T \left(\frac{c}{D_T}\right)_4, \quad B = I, \quad \text{or} \quad \wp(\frac{c\ \omega}{D}) \ .$$

Since

$$v_2((\wp(\frac{c\ \omega}{D}))^2 + 1) = v_2(\wp(\frac{c\ \omega}{D}) - I) + v_2(\wp(\frac{c\ \omega}{D}) + I) = \frac{3}{4} + \frac{3}{4} = \frac{3}{2} \ ,$$

so

$$v_2\left(\frac{2B}{(\wp(\frac{c\ \omega}{D}))^2+1}\right)=1-\frac{3}{2}=-\frac{1}{2} \qquad \text{(and obviously we have } v_2(B)=0) \ ,$$

therefore we have

$$v_2(S^*(D))\geq -\frac{1}{2}+v_2\left(\sum_T\left(\frac{c}{D_T}\right)_4\right)\geq -\frac{1}{2}+\frac{n}{2}=\frac{n-1}{2}.$$

This proves the lemma.

Proof of Theorem 1.2 . First let us prove

$$v_2\left(L(\overline{\psi}_D,1)/\omega\right)\geq \frac{n-1}{2}.$$

Take sums for both sides of formula (1.1) over all subsets T of $\{1,\cdots,n\}$, we have

$$\sum_T\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)=\frac{I}{2}\sum_T\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4\frac{1}{\wp(\frac{c\ \omega}{D})-I}+\frac{1}{4}\sum_T\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \quad .$$

so By Lemma 2.2 and (1.2) , we obtain

$$\sum_T\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)=S^*(D)+\frac{\sharp\mathcal{C}}{4}. \tag{2.9}$$

Also we have

$$v_2(\frac{\sharp\mathcal{C}}{4})=v_2\left(\frac{\prod\limits_{k=1}^{n}(\pi_k\overline{\pi}_k-1)}{4}\right)\geq 3n-2\geq n \ ,$$

so By Lemma 2.4 we obtain

$$v_2\left(\sum_T\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)\right)\geq\frac{n-1}{2} \quad .$$

By Lemma 2.1 we know $L_S(\overline{\psi}_{D_T},1)=L(\overline{\psi}_D,1)$ when $T=\{1,\cdots,n\}$; and when $T=\emptyset$ we have

$$L_S(\overline{\psi}_{D_T},1)=L_S(\overline{\psi}_1,1)=L(\overline{\psi}_1,1)\prod_{k=1}^{n}(1-\frac{\overline{\pi}_k}{\pi_k\overline{\pi}_k})$$

$$=L(\overline{\psi}_1,1)\prod_{k=1}^{n}(1-\frac{1}{\pi_k}).$$

By [B-SD] or [Zhao] we know $L(\overline{\psi}_1,1)=\omega/4$, so

$$L_S(\overline{\psi}_1,1)=\frac{\omega}{4}\prod_{k=1}^{n}(1-\frac{1}{\pi_k}),$$

$$v_2\left(L_S(\overline{\psi}_1,1)/\omega\right)=v_2\left(\frac{1}{4}\prod_{k=1}^{n}(1-\frac{1}{\pi_k})\right)\geq 2n-2 \qquad (\text{since} \quad v_2(\pi_k-1)\geq 2).$$

Now we use induction on n to prove our assertion $v_2\left(L(\overline{\psi}_D,1)/\omega\right)\geq\frac{n-1}{2}$. If $n=1$, then $D=\pi_1$, $L_S(\overline{\psi}_1,1)=(\omega/4)\cdot(\pi_1-1)/\pi_1$. Since $\pi_1\equiv 1(mod\ 4)$, so $v_2\left(L_S(\overline{\psi}_1,1)/\omega\right)\geq 0$. By the above analysis we have

$$v_2\left(\frac{\pi_1}{\omega}\left(\frac{\theta}{1}\right)_4 L_S(\overline{\psi}_1,1)+\frac{\pi_1}{\omega}\overline{\left(\frac{\theta}{\pi_1}\right)_4}L_S(\overline{\psi}_{\pi_1},1)\right)\geq\frac{1-1}{2}=0\ ,$$

therefore we have

$$v_2\left(L(\overline{\psi}_{\pi_1},1)/\omega\right)=v_2\left(L_S(\overline{\psi}_{\pi_1},1)/\omega\right)\geq 0 \quad .$$

Now assume our assertion is true for cases $1,2,\cdots,n-1$, and consider the case n, $\quad D=\pi_1\cdots\pi_n$. For any subset T of $\{1,\cdots,n\}$, denote $t=t(T)=\sharp T$, by Lemma 2.1 we know

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)=\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L(\overline{\psi}_{D_T},1)\prod_{\pi_k|\widehat{D}_T}\left(1-\left(\frac{D_T}{\pi_k}\right)_4\frac{1}{\pi_k}\right)$$

Since $(D_T/\pi_k)_4=\pm 1,\pm I$, so

$$1-\left(\frac{D_T}{\pi_k}\right)_4\frac{1}{\pi_k}=\frac{\pi_k-\mu}{\pi_k} \qquad \mu\in\{\pm 1,\pm I\}.$$

Note that $\pi_k\equiv 1(mod\ 4)$, so we know $\quad v_2(\pi_k-\mu)\geq 1/2$; moreover the equality holds if and only if $(D_T/\pi_k)_4^2=-1$. Thus by our inductive assumption, we have

$$\begin{aligned}v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)\right)&=v_2\left(L(\overline{\psi}_{D_T},1)/\omega\right)+\sum_{\pi_k|\widehat{D}_T}v_2\left(1-\left(\frac{D_T}{\pi_k}\right)_4\frac{1}{\pi_k}\right)\\&\geq\frac{t-1}{2}+\frac{1}{2}\cdot\sharp\{\pi_k\ :\ \pi_k|\widehat{D}_T\}=\frac{t-1}{2}+\frac{n-t}{2}=\frac{n-1}{2}.\end{aligned}$$

Also when $T=\emptyset$ we have

$$L_S(\overline{\psi}_{D_T},1)=L_S(\overline{\psi}_1,1)=L(\overline{\psi}_1,1)\prod_{k=1}^{n}(1-\frac{1}{\pi_k})=\frac{\omega}{4}\prod_{k=1}^{n}(1-\frac{1}{\pi_k}),$$

therefore

$$v_2\left(L_S(\overline{\psi}_1,1)/\omega\right)\geq 2n-2\geq\frac{n-1}{2}\ ,$$

$$\begin{aligned}v_2\left(L(\overline{\psi}_D,1)/\omega\right)&=v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D}\right)_4}L(\overline{\psi}_D,1)\right)\\&=v_2\left(\sum_T\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)-\sum_{\emptyset\neq T\subsetneqq\{1,\cdots,n\}}\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)-\frac{D}{\omega}L_S(\overline{\psi}_1,1)\right)\\&\geq\frac{n-1}{2}.\end{aligned}$$

Therefore by induction we have proved our assertion that $v_2\left(L(\overline{\psi}_D,1)/\omega\right) \geq \frac{n-1}{2}$ holds for any positive integer n.

Now we consider the condition for the equality holds, using induction method on n too. If $n=1$, then $D=\pi_1$, by (2.9) we obtain

$$\frac{\pi_1}{\omega}\overline{\left(\frac{\theta}{1}\right)_4}L_{\pi_1}(\overline{\psi}_1,1)+\frac{\pi_1}{\omega}\overline{\left(\frac{\theta}{\pi_1}\right)_4}L_{\pi_1}(\overline{\psi}_{\pi_1},1)=S^*(\pi_1)+\frac{\pi_1\overline{\pi_1}-1}{4}\ ,$$

that is

$$\frac{1}{4}(\pi_1-1)+\frac{\pi_1}{\omega}\left(\frac{\theta}{\pi_1}\right)_4 L(\overline{\psi}_{\pi_1},1)=S^*(\pi_1)+\frac{\pi_1\overline{\pi_1}-1}{4}\ .$$

Since

$$v_2\left(\frac{\pi_1\overline{\pi_1}-1}{4}\right)=v_2(\pi_1\overline{\pi_1}-1)-2\geq 1,\quad v_2(S^*(\pi_1))\geq\frac{1-1}{2}=0\quad \text{(Lemma 2.4)},$$

so the equality

$$\begin{aligned}v_2\left(L(\overline{\psi}_{\pi_1},1)/\omega\right)&=v_2\left(\frac{\pi_1}{\omega}\overline{\left(\frac{\theta}{\pi_1}\right)_4}L_{\pi_1}(\overline{\psi}_{\pi_1},1)\right)\\&=v_2\left(S^*(\pi_1)+\frac{\pi_1\overline{\pi_1}-1}{4}-\frac{1}{4}(\pi_1-1)\right)\\&=0\end{aligned}$$

holds if and only if one of the following conditions is true:

(1) $v_2(\pi_1-1)=2$ when $v_2(S^*(\pi_1))>0$;

(2) $v_2(\pi_1-1)>2$ when $v_2(S^*(\pi_1))=0$.

Thus we know

$$v_2\left(L(\overline{\psi}_{\pi_1},1)/\omega\right)=0 \text{ holds if and only if } \delta_1(\pi_1)=s_1(\pi_1)+\varepsilon_1(\pi_1)=1.$$

Assume our result is true in the cases $1,\cdots,n-1$, consider the case n, i.e. $D=\pi_1\cdots\pi_n$. When $T=\emptyset$, we have

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)=\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_D(\overline{\psi}_1,1)=\frac{D}{\omega}L(\overline{\psi}_1,1)\prod_{k=1}^{n}(1-\frac{1}{\pi_k}),$$

$$\begin{aligned}v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)\right)&=v_2\left(L(\overline{\psi}_1,1)/\omega\right)+\sum_{k=1}^{n}v_2(\pi_k-1)\\&=v_2(\frac{1}{4})+\sum_{k=1}^{n}v_2(\pi_k-1)\\&\geq 2n-2\geq n\geq\frac{n-1}{2}\ .\end{aligned}$$

When $\emptyset \neq T \subsetneq \{1, \cdots, n\}$ we have ,

$$\begin{aligned} v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1)\right) &= v_2\left(L_S(\overline{\psi}_{D_T}, 1)/\omega\right) \\ &= v_2\left(\frac{L(\overline{\psi}_{D_T}, 1)}{\omega} \cdot \prod_{\pi_k | \widehat{D}_T}\left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k}\right)\right) \\ &= v_2\left(L(\overline{\psi}_{D_T}, 1)/\omega\right) + \sum_{\pi_k | \widehat{D}_T} v_2\left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k}\right). \end{aligned}$$

Since $(D_T/\pi_k)_4 = \pm 1, \ \pm I$, so

$$1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k} = \frac{\pi_k - \mu}{\pi_k} = \frac{\pi_k - 1 + (1-\mu)}{\pi_k}, \quad \mu \in \{\pm 1, \pm I\}.$$

Therefore $v_2\left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k}\right) \geq \frac{1}{2}$, and the equality holds if and only if $(D_T/\pi_k)_4 = \pm I$, i.e. $(D_T/\pi_k)_4^2 = -1$, that is $\left[\frac{D_T}{\pi_k}\right]_2 = 1$. Thus

$$v_2\left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k}\right) = \frac{1}{2} \quad \text{if and only if} \quad \left[\frac{D_T}{\pi_k}\right]_2 = 1.$$

By the proof of the first part of the Theorem we know

$$v_2\left(L(\overline{\psi}_{D_T}, 1)/\omega\right) \geq \frac{t(T)-1}{2}, \quad t(T) = \sharp T\ ,$$

and by our inductive assumption we know the equality holds if and only if $\delta_t(D_T) = 1, \quad t = t(T)$. Thus

$$v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1)\right) \geq \frac{t(T)-1}{2} + \frac{n - t(T)}{2} = \frac{n-1}{2},$$

and the equality holds if and only if $\left[\frac{D_T}{\pi_k}\right]_2 = 1$ (for any $\pi_k | \widehat{D}_T$) and $\delta_t(D_T) = 1$. That is to say the equality

$$v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1)\right) = \frac{n-1}{2}$$

holds if and only if

$$\left(\prod_{\pi_k | \widehat{D}_T}\left[\frac{D_T}{\pi_k}\right]_2\right)\delta_t(D_T) = 1.$$

For the elliptic curve $E_{D_T}: \ y^2 = x^3 - D_T x$ and Hecke characters ψ_{D_T}, by [Ru 1-2] we know $L(\overline{\psi}_{D_T}, 1)/\Omega \in K = \mathbb{Q}(I)$, also we have $\Omega = \frac{\omega}{\sqrt[4]{D_T}}$, so

$$\begin{aligned} L(\overline{\psi}_{D_T}, 1)/\omega &= (\sqrt[4]{D_T})^{-1} \cdot L(\overline{\psi}_{D_T}, 1)/\frac{\omega}{\sqrt[4]{D_T}} \\ &= (\sqrt[4]{D_T})^{-1} \cdot L(\overline{\psi}_{D_T}, 1)/\Omega \in K(\sqrt[4]{D_T}), \end{aligned}$$

i.e. $L(\overline{\psi}_{D_T},1)/\omega \in K(\sqrt[4]{D_T})$. Thus by Lemma 2.1 we know

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1) = D\overline{\left(\frac{\theta}{D_T}\right)_4}\prod_{\pi_k|\widehat{D}_T}\left(1-\left(\frac{D_T}{\pi_K}\right)_4\frac{1}{\pi_k}\right)\cdot L(\overline{\psi}_{D_T},1)/\omega \in K(\sqrt[4]{D_T}),$$

and if $v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)\right) = \frac{n-1}{2}$, then we have

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1) = (1+I)^{n-1}\alpha_T\sqrt[4]{D_T^3},$$

where $\alpha_T \in K$, and $v_2(\alpha_T)=0$. (since $v_2(\sqrt[4]{D_T}^3) = \frac{3}{4}v_2(D_T)=0$). For any subsets T and T' of $\{1,\cdots,n\}$, if $v_2(\alpha_T) = v_2(\alpha_{T'})=0$, then it could be easily verified that

$$v_2\left(\alpha_T\sqrt[4]{D_T^3}+\alpha_{T'}\sqrt[4]{D_{T'}^3}\right)>0 \quad .$$

Thus, consider the terms in the sum

$$\sum_{\emptyset\neq T\subsetneqq\{1,\cdots,n\}}\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1),$$

for any two terms with 2-adic valuations equal to $(n-1)/2$, the 2-adic valuation of their sum would be bigger than $(n-1)/2$. So when $n>1$ we have

$$v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_\emptyset}\right)_4}L_S(\overline{\psi}_{D_\emptyset},1)\right)\geq 2n-2\geq n>\frac{n-1}{2}.$$

Hence we know $v_2\left(L(\overline{\psi}_D,1)/\omega\right) = \frac{n-1}{2}$ holds if and only if one of the following statements is true :

(1) $v_2(S^*(D)) > (n-1)/2$;

(2) $v_2(S^*(D)) = \frac{n-1}{2}$.

Statement (1) means that, when $\varepsilon_n(D)=0$, in the sum

$$\sum_{\emptyset\neq T\subsetneqq\{1,\cdots,n\}}\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1),$$

the number of terms with 2-adic valuation $(n-1)/2$ is odd. The number of such terms turns to be

$$\begin{aligned}
&\sharp\left\{\emptyset\neq T\subsetneqq\{1,\cdots,n\}:\ v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)\right)=\frac{n-1}{2}\right\}\\
&=\sharp\left\{\emptyset\neq T\subsetneqq\{1,\cdots,n\}:\ \left(\prod_{\pi_k|\widehat{D}_T}\left[\frac{D_T}{\pi_k}\right]_2\right)\delta_t(D_T)=1\right\}\\
&\equiv\sum_{\emptyset\neq T\subsetneqq\{1,\cdots,n\}}\left(\prod_{\pi_k|\widehat{D}_T}\left[\frac{D_T}{\pi_k}\right]_2\right)\delta_t(D_T)\\
&\equiv 1(mod\ 2),
\end{aligned}$$

we have

$$\delta_n(D) = \varepsilon_n(D) + \sum_{\emptyset \neq T \subsetneqq \{1,\cdots,n\}} \left(\prod_{\pi_k | \widehat{D}_T} \left[\frac{D_T}{\pi_k} \right]_2 \right) \delta_t(D_T) \equiv 1 (mod\ 2).$$

On the other hand, the statement (2) above means that, when $\varepsilon_n(D) = 1$, in the sum

$$\sum_{\emptyset \neq T \subsetneqq \{1,\cdots,n\}} \frac{D}{\omega} \overline{\left(\frac{\theta}{D_T} \right)_4} L_S(\overline{\psi}_{D_T}, 1),$$

the number of terms with 2-adic valuation $(n-1)/2$ is even, that is

$$\sum_{\emptyset \neq T \subsetneqq \{1,\cdots,n\}} \left(\prod_{\pi_k | \widehat{D}_T} \left[\frac{D_T}{\pi_k} \right]_2 \right) \delta_t(D_T) \equiv 0 (mod\ 2).$$

We have

$$\delta_n(D) = \varepsilon_n(D) + \sum_{\emptyset \neq T \subsetneqq \{1,\cdots,n\}} \left(\prod_{\pi_k | \widehat{D}_T} \left[\frac{D_T}{\pi_k} \right]_2 \right) \delta_t(D_T) \equiv 1 (mod\ 2).$$

By the discussion above we know that $v_2\left(L(\overline{\psi}_D, 1)/\omega\right) = (n-1)/2$ if and only if $\delta_n(D) = 1$. This proves the theorem.

Proof of Theorem 1.3. This theorem follows from above Theorem 1.2 and the main result of Coates-Wiles in [Co-Wi] .

For elliptic curve $E_D : y^2 = x^3 - Dx$, with $D = \pi_1^2 \cdots \pi_r^2 \pi_{r+1} \cdots \pi_n$, where $\pi_k \equiv 1 (mod\ 4)$ are distinct Gaussian prime integers ($k = 1, \cdots, n$), we could prove Theorem 1.4 and Theorem 1.5 similarly as proving Theorem 1.1 and 1.2.

III. 3-Valuations of $L-$series of Elliptic Curves with CM by $\sqrt{-3}$

Now assume the number field $K = \mathbb{Q}(\sqrt{-3})$, $\tau = (-1+\sqrt{-3})/2 = exp(2\pi I/3)$ is a primitive cubic root of unity, $O_K = \mathbb{Z}[\tau]$ is the ring of integers of K, $I = \sqrt{-1}$. We now study elliptic curves with complex multiplication by $\sqrt{-3}$ and prove Theorem 1.6 and 1.7.

Consider elliptic curve $E = E_{D^2} : y^2 = x^3 + D$, $D \in O_K$. Let $\psi_{E/K}$ be the Hecke character of $K = \mathbb{Q}(\sqrt{-3})$ corresponding to the elliptic curve E/K (The discriminant of E is $\Delta(E) = -2^4 3^3 D^2$). For any prime ideal $\wp \dagger 6D$ of K, put

$$\mathcal{A}_\wp = N_{K/\mathbb{Q}}(\wp) + 1 - \sharp \widetilde{E}(\mathbb{F}_\wp),$$

where $\mathbb{F}_\wp = O_K/\wp$ is the residue field of K modulo $\wp$, the symbol " $N_{K/\mathbb{Q}}(\cdot)$" (or " $N(\cdot)$") denote the norm map of ideals from K to $\mathbb{Q}$, $\widetilde{E}$ is the reduction curve of the elliptic curve E modulo $\wp$. Then by [Sil 2] and [Sil 1] , we have

$$N_{K/\mathbb{Q}}(\wp) = N_{K/\mathbb{Q}}(\psi_{E/K}(\wp)),$$

$$\sharp\widetilde{E}(\mathbb{F}_\wp) = N_{K/\mathbb{Q}}(\wp) + 1 - \psi_{E/K}(\wp) - \overline{\psi_{E/K}(\wp)},$$

$$\mathcal{A}_\wp = \psi_{E/K}(\wp) + \overline{\psi_{E/K}(\wp)}.$$

By definition, we know that the $L-$series E_D (omitting Euler factors corresponding to bad reductions)is

$$\begin{aligned}
L_D(s) &= \prod_{\wp\dagger 6D} (1 - \mathcal{A}_\wp N(\wp)^{-s} + N(\wp)^{1-2s})^{-1} \\
&= \prod_{\wp\dagger 6D} (1 + (\sharp\widetilde{E}(\mathbb{F}_\wp) - N(\wp) - 1)N(\wp)^{-s} + N(\wp)^{1-2s})^{-1} \\
&= \prod_{\wp\dagger 6D} (1 - (\psi_{E/K}(\wp) + \overline{\psi_{E/K}(\wp)})N(\wp)^{-s} + N(\wp)^{1-2s})^{-1} \\
&= \prod_{\wp\dagger 6D} (1 - (\psi_{E/K}(\wp)N(\wp)^{-s})^{-1} \cdot \prod_{\wp\dagger 6D} (1 - \overline{\psi_{E/K}(\wp)}N(\wp)^{-s})^{-1}
\end{aligned}$$

Also by [Sil 2] we know ,

$$\psi_{E/K}(\wp) = \overline{\left(\frac{4D}{\pi}\right)_6}\pi,$$

where $\wp = (\pi)$and $\pi \equiv 1(mod\ 3)$ is a primitive prime element . So

$$\begin{aligned}
L_D(s) = \prod_{\text{prime } \pi\equiv 1(mod\ 3)} & \left(1 - \left(\overline{\left(\frac{4D}{\pi}\right)_6}\pi + \left(\frac{4D}{\pi}\right)_6\overline{\pi}\right) N(\pi)^{-s} + N(\pi)^{1-2s}\right)^{-1} \\
= \prod_{\pi\equiv 1(mod\ 3)} & \left[\left(1 - \overline{\left(\frac{4D}{\pi}\right)_6} \cdot \frac{\pi}{(N(\pi))^s}\right)^{-1} \cdot \left(1 - \left(\frac{4D}{\pi}\right)_6 \cdot \frac{\overline{\pi}}{(N(\pi))^s}\right)^{-1}\right] \\
= L(s, \psi_{E/K}) & \cdot L(s, \overline{\psi}_{E/K}) \quad .
\end{aligned}$$

Thus, ignoring Euler factors corresponding to primes $\wp|6D$, we have

$$L_D(s) = L(s, \psi_{E/K}) \cdot L(s, \overline{\psi}_{E/K}). \tag{3.1}$$

Now assume D is a rational integer , Consider the elliptic curve $E_D:\ y^2 = x^3 + D,\ D \in \mathbb{Z}$. For any rational prime $p\dagger 6D$, E_D has good reduction at p (Since the discriminant of E_D is $-2^4 3^3 D^2$). Let $N_p = \sharp\widetilde{E}_D(\mathbb{F}_p)$, i.e. the number of $\mathbb{F}_p-$rational points of the reduced curve $\widetilde{E}_D$. We know

(1) when $p \equiv 2(mod\ 3)$, then $N_p = p+1$;

(2) when $p \equiv 1(mod\ 3)$, then $N_p = p+1-\left(\frac{4D}{\overline{\pi}}\right)_6 \pi - \left(\frac{4D}{\pi}\right)_6 \overline{\pi}$,

where $p = \pi\overline{\pi} = N_{K/\mathbb{Q}}(\pi)$, $\pi \in \mathbb{Z}[\tau]$ and $\pi \equiv 1(mod\ 3)$ is a prime element (see [Ire-Ro]). Thus

$$L_D(s) = \prod_{p\equiv 2(mod\ 3)} (1+p^{1-2s})^{-1} \cdot \prod_{p\equiv 1(mod\ 3)} \left(1 - \pi\left(\frac{4D}{\overline{\pi}}\right)_6 p^{-s} - \overline{\pi}\left(\frac{4D}{\pi}\right)_6 p^{-s} + p^{1-2s}\right)^{-1}$$

$$= \prod_{\text{prime } \pi\equiv 1(mod\ 3)} \left(1 - \left(\frac{4D}{\pi}\right)_6 \cdot \frac{\overline{\pi}}{(N(\pi))^s}\right)^{-1} .$$

where the product $\prod$ is taken over primitive prime elements prime $\pi(\equiv 1(mod\ 3))$ of $K = \mathbb{Q}(\sqrt{-3})$. So

$$L_D(s) = \sum_{\sigma\equiv 1(mod\ 3)} \left(\frac{4D}{\sigma}\right)_6 \cdot \frac{\overline{\sigma}}{(N(\sigma))^s} . \qquad (3.2)$$

where the sum Σ is taken over primitive integers $\sigma(\equiv 1(mod\ 3))$ in K.

Remark 3.1 . The $L_D(s)$ in (3.1) denotes the $L-$series of E/K , i.e. $L_D(s) = L(E/K, s)$. While the $L_D(s)$ in (3.2) denotes the $L-$series of $E/\mathbb{Q}$, i.e. $L_D(s) = L(E/\mathbb{Q}, s)$. So if D is a rational integer, by these two formulae we obtain $L(E/K, s) = (L(E/\mathbb{Q}, s))^2$ (a square), and

$$L(E/\mathbb{Q}, s) = L(s, \psi_{E/K}) = L(s, \overline{\psi}_{E/K})$$

(ignoring Euler factors corresponding to prime$\wp|6D$).

Now assume $D \in O_K = \mathbb{Z}[\tau]$, and $D \equiv 1(mod\ 3)$ is a primitive integer , Δ is the square-free part of D, i.e. the product of all distinct prime divisors of D, and $\Delta \equiv 1(mod\ 3)$. Assume $L_D(s)$ as in (3.2). Then For elliptic curve $E_D:\ y^2 = x^3 + 2^4 D$, we have

$$L_D(s) = \sum_{\sigma\equiv 1(mod\ 3)} \left(\frac{D}{\sigma}\right)_6 \cdot \frac{\overline{\sigma}}{(N(\sigma))^s}.$$

Since $(3, \Delta) = 1$, so $\sigma = v \cdot 3\Delta + 3\beta + \Delta$, where v is an algebraic integer in $K = \mathbb{Q}(\sqrt{-3})$, β runs over a residue system of O_K modulo Δ. By the cubic reciprocity we obtain

$$\left(\frac{D}{\sigma}\right)_6^2 = \left(\frac{D}{\sigma}\right)_3 = \left(\frac{\sigma}{D}\right)_3 = \left(\frac{v\cdot 3\Delta + 3\beta + \Delta}{D}\right)_3 = \left(\frac{3\beta}{D}\right)_3 = \left(\frac{3\beta}{D}\right)_6^2 ,$$

so $\quad \left(\frac{D}{\sigma}\right)_6 = \pm\left(\frac{3\beta}{D}\right)_6$, that means the value of $\left(\frac{D}{\sigma}\right)_6$ depends only on β, not on v. Thus

$$\begin{aligned}
L_D(s) &= \sum_{\sigma\equiv 1(mod\ 3)} \left(\frac{D}{\sigma}\right)_6 \cdot \frac{\overline{\sigma}}{(N(\sigma))^s} \\
&= \sum_{\sigma\equiv 1(mod\ 3)} \left(\frac{D}{\sigma}\right)_6 \cdot \frac{3\overline{v}\overline{\Delta}+3\overline{\beta}+\overline{\Delta}}{N(3v\Delta+3\beta+\Delta)^s} \\
&= \sum_{\beta} \left(\frac{D}{\sigma}\right)_6 \sum_{v} \frac{3\overline{v}\overline{\Delta}+3\overline{\beta}+\overline{\Delta}}{N(3v\Delta+3\beta+\Delta)^s} \\
&= \sum_{\beta} \left(\pm\left(\frac{3\beta}{D}\right)_6\right) \sum_{v} \frac{3\overline{v}\overline{\Delta}+3\overline{\beta}+\overline{\Delta}}{N(3v\Delta+3\beta+\Delta)^s}.
\end{aligned}$$

So we have

$$L_D(s) = \sum_{\beta} \left(\pm\left(\frac{3\beta}{D}\right)_6\right) \sum_{v} \frac{3\overline{v}\overline{\Delta}+3\overline{\beta}+\overline{\Delta}}{N(3v\Delta+3\beta+\Delta)^s} \qquad (3.3)$$

The function $L_D(s)$ defined by these formula is convergent for $Re(s) > \frac{3}{2}$.

Now consider the analytic extension of the above $L-$series $L_D(s)$. we could analytically extend it near to $s=1$, and express it as a finite sum. First, for the period lattice $L = O_K = \mathbb{Z}+\mathbb{Z}\tau$ and $z \in \mathbb{C}-L$, Define the following series as in [Go-Sch] :

$$\psi(z,s,L) = \frac{\overline{z}}{|z|^{2s}} + \sum_{\alpha\in L-\{0\}} \left\{ \frac{\overline{z}+\overline{\alpha}}{|z+\alpha|^{2s}} - \frac{\overline{\alpha}}{|\alpha|^{2s}}\left(1-\frac{sz}{\alpha}-\frac{(s-1)\overline{z}}{\overline{\alpha}}\right)\right\} \qquad (3.4)$$

This series is convergent and defines an analytic function for $\quad Re(s) > \frac{1}{2}$, , and $\psi(z,s,L)$ is uniformly convergent to $\zeta(z,L)$ (the Weierstrass $\quad Zeta$function over the period lattice L) when $s \to 1$, that is

$$\begin{aligned}
\psi(z,1,L) &= \frac{\overline{z}}{|z|^2} + \sum_{\alpha\in L-\{0\}} \left\{ \frac{\overline{z}+\overline{\alpha}}{|z+\alpha|^2} - \frac{\overline{\alpha}}{|\alpha|^2}(1-\frac{z}{\alpha})\right\} \\
&= \frac{1}{z} + \sum_{\alpha\in L-\{0\}} \left\{ \frac{1}{z+\alpha} - \frac{1}{\alpha} + \frac{z}{\alpha^2}\right\} \\
&= \zeta(z,L).
\end{aligned}$$

Also when $Re(s)$ is sufficiently large, we know that

$$\sum_{\alpha\in L-\{0\}} \frac{\overline{\alpha}}{|\alpha|^{2s}} \qquad \text{and} \qquad \sum_{\alpha\in L-\{0\}} \frac{1}{|\alpha|^{2s}}\cdot\frac{\overline{\alpha}}{\alpha}$$

are absolutely convergent, so their terms may be re-arranged. Now we show that when $Re(s)$ is sufficiently large we have

$$\sum_{\alpha\in L-\{0\}} \frac{\overline{\alpha}}{|\alpha|^{2s}} = \sum_{\alpha\in L-\{0\}} \frac{1}{|\alpha|^{2s}}\cdot\frac{\overline{\alpha}}{\alpha} = 0 \qquad (3.5)$$

In fact, we know that the unit group of O_K is $\{\pm 1, \pm\tau, \pm\tau^2\}$, and the series is absolutely convergent, so in the summation over the integers α, we could first add all the terms corresponding to the associates $\pm\alpha, \pm\tau\alpha, \pm\tau^2\alpha$ of an integer α. Obviously $|\mu\alpha|^{2s} = |\alpha|^{2s}$ (for any unit μ of K). Since

$$\overline{-\alpha} = -\overline{\alpha}, \quad \overline{-\tau\alpha} = -\overline{\tau\alpha}, \quad \overline{-\tau^2\alpha} = -\overline{\tau^2\alpha},$$

so

$$\frac{\overline{\alpha}}{|\alpha|^{2s}} + \frac{\overline{-\alpha}}{|-\alpha|^{2s}} + \frac{\overline{\tau\alpha}}{|\tau\alpha|^{2s}} + \frac{\overline{-\tau\alpha}}{|-\tau\alpha|^{2s}} + \frac{\overline{\tau^2\alpha}}{|\tau^2\alpha|^{2s}} + \frac{\overline{-\tau^2\alpha}}{|-\tau^2\alpha|^{2s}} = 0.$$

Therefore we have

$$\sum_{\alpha\in L-\{0\}} \frac{\overline{\alpha}}{|\alpha|^{2s}} = 0.$$

Similarly, by

$$\frac{\overline{-\alpha}}{-\alpha} = \frac{\overline{\alpha}}{\alpha} \qquad \frac{\overline{\tau\alpha}}{\tau\alpha} = \tau\cdot\frac{\overline{\alpha}}{\alpha} \qquad \frac{\overline{-\tau\alpha}}{-\tau\alpha} = \tau\cdot\frac{\overline{\alpha}}{\alpha},$$
$$\frac{\overline{\tau^2\alpha}}{\tau^2\alpha} = \tau^2\cdot\frac{\overline{\alpha}}{\alpha} \qquad \frac{\overline{-\tau^2\alpha}}{-\tau^2\alpha} = \tau^2\cdot\frac{\overline{\alpha}}{\alpha},$$

and $\quad 1+\tau+\tau^2 = 0$, so we know

$$\frac{\overline{-\alpha}}{-\alpha} + \frac{\overline{\alpha}}{\alpha} + \frac{\overline{\tau\alpha}}{\tau\alpha} + \frac{\overline{-\tau\alpha}}{-\tau\alpha} + \frac{\overline{\tau^2\alpha}}{\tau^2\alpha} + \frac{\overline{-\tau^2\alpha}}{-\tau^2\alpha} = 2(1+\tau+\tau^2)\cdot\frac{\overline{\alpha}}{\alpha} = 0,$$

Thus we have

$$\sum_{\alpha\in L-\{0\}} \frac{1}{|\alpha|^{2s}}\cdot\frac{\overline{\alpha}}{\alpha} = 0.$$

Therefore, when $Re(s)$ is sufficiently large, by the series (3.4), defined above we have

$$\begin{aligned}\sum_{\alpha\in L} \frac{\overline{z}+\overline{\alpha}}{|z+\alpha|^{2s}} &= \psi(z,s,L) + \sum_{\alpha\in L-\{0\}} \frac{\overline{\alpha}}{|\alpha|^{2s}} - sz\sum_{\alpha\in L-\{0\}} \frac{1}{|\alpha|^{2s}}\cdot\frac{\overline{\alpha}}{\alpha} + (1-s)\sum_{\alpha\in L-\{0\}} \frac{\overline{z}}{|\alpha|^{2s}} \\ &= \psi(z,s,L) + (1-s)\overline{z}\sum_{\alpha\in L-\{0\}} \frac{1}{(N(\alpha))^s}.\end{aligned}$$

That is

$$\sum_{\alpha\in L} \frac{\overline{z}+\overline{\alpha}}{|z+\alpha|^{2s}} = \psi(z,s,L) + 6\overline{z}(1-s)\zeta_K(s) \qquad (3.6)$$

where $\zeta_K(s)$is the Dedekind Zeta-function of the number field $K = \mathbb{Q}(\sqrt{-3})$ Since $\psi(z,s,L)$ is an analytic function in the area $\quad Re(s) > \frac{1}{2}$, and $\zeta_K(s)$ is an analytic function for $\quad Re(s) > 1$, so the right side of formula (3.6) gives an analytic extension for the series $\quad \sum_{\alpha\in L} \frac{\overline{z}+\overline{\alpha}}{|z+\alpha|^{2s}}$ $\quad$ (to the

area $Re(s) > 1$). Now transform the right side of formula (3.3) as follows:

$$\sum_v \frac{3\overline{v}\overline{\Delta} + 3\overline{\beta} + \overline{\Delta}}{N(3v\Delta + 3\beta + \Delta)^s} = \sum_v \frac{3\overline{\Delta}\left(\overline{v} + \frac{3\overline{\beta}+\overline{\Delta}}{3\overline{\Delta}}\right)}{(N(3\Delta))^s\left(N\left(v + \frac{3\beta+\Delta}{3\Delta}\right)\right)^s}$$

$$= \frac{3\overline{\Delta}}{N(3\Delta)^s}\sum_v \frac{\overline{v} + \frac{3\overline{\beta}+\overline{\Delta}}{3\overline{\Delta}}}{N\left(v + \frac{3\beta+\Delta}{3\Delta}\right)^s} \quad .$$

By the analytic extension of (3.6) we could obtain the analytic extension of

$$\sum_v \frac{\overline{v} + \frac{3\overline{\beta}+\overline{\Delta}}{3\overline{\Delta}}}{N\left(v + \frac{3\beta+\Delta}{3\Delta}\right)^s},$$

and hence obtain the analytic extension of $L_D(s)$ as the following:

$$L_D(s) = \sum_\beta \left(\pm\left(\frac{3\beta}{D}\right)_6\right)\sum_v \frac{3\overline{v}\overline{\Delta} + 3\overline{\beta} + \overline{\Delta}}{N(3v\Delta + 3\beta + \Delta)^s}$$

$$= \sum_\beta \left(\pm\left(\frac{3\beta}{D}\right)_6\right)\cdot\frac{3\overline{\Delta}}{N(3\Delta)^s}\cdot\sum_v \frac{\overline{v} + \frac{3\overline{\beta}+\overline{\Delta}}{3\overline{\Delta}}}{N\left(v + \frac{3\beta+\Delta}{3\Delta}\right)^s}$$

$$= \frac{3\overline{\Delta}}{N(3\Delta)^s}\cdot\sum_\beta \left(\pm\left(\frac{3\beta}{D}\right)_6\right)\left[\psi(\frac{\beta}{\Delta} + \frac{1}{3}, s, L) + 6(\frac{\overline{\beta}}{\overline{\Delta}} + \frac{1}{3})(1-s)\zeta_K(s)\right].$$

By class number formula of imaginary quadratic field (see e.g. [Wa], [Zhang]) we obtain

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2\pi}{6\sqrt{3}}\cdot h(K) = \frac{2\pi}{6\sqrt{3}} = \frac{\pi}{3\sqrt{3}}.$$

Thus let $s \to 1$, we obtain

$$L_D(1) = \frac{3\overline{\Delta}}{N(3\Delta)}\sum_\beta \left(\pm\left(\frac{3\beta}{D}\right)_6\right)\left[\psi(\frac{\beta}{\Delta} + \frac{1}{3}, 1, L) - \frac{2\pi}{\sqrt{3}}(\frac{\overline{\beta}}{\overline{\Delta}} + \frac{1}{3})\right].$$

Also we have $\psi(z, 1, L) = \zeta(z, L)$, therefore

$$L_D(1) = \frac{1}{3\Delta}\sum_\beta \left(\pm\left(\frac{3\beta}{D}\right)_6\right)\left[\zeta(\frac{\beta}{\Delta} + \frac{1}{3}, L) - \frac{2\pi}{\sqrt{3}}(\frac{\overline{\beta}}{\overline{\Delta}} + \frac{1}{3})\right] \qquad (3.7)$$

By results in [Ste] we know , the Weiestrass $\wp-$function corresponding to the period lattice $L = O_K = \mathbb{Z} + \mathbb{Z}_\tau$ is $\Omega^2\wp(\Omega z)$, where function $\wp(z)$ satisfies $\wp'(z)^2 = 4\wp(z)^3 - 1$, and the corresponding period lattice is ΩO_K ($\Omega = 3.059908\cdots$). And we have the following formula :

$$\zeta(\alpha_1 + \alpha_2) = \zeta(\alpha_1) + \zeta(\alpha_2) + \frac{\Omega}{2}\cdot\frac{\wp'(\Omega\alpha_1) - \wp'(\Omega\alpha_2)}{\wp(\Omega\alpha_1) - \wp(\Omega\alpha_2)}. \qquad (3.8)$$

Thus

$$L_D(1)=\frac{1}{3\Delta}\sum_{\beta}\left(\pm\left(\frac{3\beta}{\Delta}\right)_6\right)\cdot$$

$$\left[\zeta(\frac{\beta}{\Delta})+\zeta(\frac{1}{3})+\frac{\Omega}{2}\cdot\frac{\wp'(\frac{\beta\Omega}{\Delta})-\wp'(\frac{\Omega}{3})}{\wp(\frac{\beta\Omega}{\Delta})-\wp(\frac{\Omega}{3})}-\frac{2\pi}{\sqrt{3}}(\frac{\overline{\beta}}{\overline{\Delta}}+\frac{1}{3})\right].$$

Also by [Ste] we know $\wp'(\frac{\Omega}{3})=-\sqrt{3},\ \wp(\frac{\Omega}{3})=1,$ so we obtain the following Proposition :

Proposition 3.1 . Let $D\equiv 1(mod\ 3)$ be a primitive integer of O_K. Then for the elliptic curve $E_D\ :\ y^2=x^3+2^4D,$ the $L-$series $L_D(s)=\sum\limits_{\sigma\equiv 1(mod\ 3)}\left(\frac{D}{\sigma}\right)_6\cdot\frac{\overline{\sigma}}{(N(\sigma))^s}$ could be analytic extended via the series $\psi(z,s,L)$ and Dedekind Zeta-function $\zeta_K(s)$, and we have

$$L_D(1)=\frac{1}{3\Delta}\sum_{\beta}\left(\pm\left(\frac{3\beta}{D}\right)_6\right)\cdot$$

$$\left[\zeta(\frac{\beta}{\Delta})+\zeta(\frac{1}{3})+\frac{\Omega}{2}\cdot\frac{\wp'(\frac{\beta\Omega}{\Delta})+\sqrt{3}}{\wp(\frac{\beta\Omega}{\Delta})-1}-\frac{2\pi}{\sqrt{3}}(\frac{\overline{\beta}}{\overline{\Delta}}+\frac{1}{3})\right]\quad(3.9)$$

Remark 3.2 . By formula (3.9) in the above Proposition 3.1 , we could in particular obtain the corresponding result in [Ste] on $L-$series $L_D(s)$ for elliptic curves $E_D\ :\ y^2=x^3-2^43^3D^2$ (with D rational integer).

Now we turn to prove Theorem 1.6 . Under the assumption of Theorem1.6 we have the following lemma by the definition of the $L-$series:

Lemma 3.1 . $$L_S(\overline{\psi}_{D_T^2},s)=\begin{cases}L(\overline{\psi}_{D_T^2},s)\ , & \text{if}\ \prod\limits_{\pi_k\in S}\pi_k=D_T\ ;\\ L(\overline{\psi}_{D_T^2},s)\prod\limits_{\pi_k|\widehat{D}_T}\left(1-\left(\frac{D_T}{\pi_k}\right)_3\cdot\frac{\overline{\pi_k}}{(\pi_k\overline{\pi_k})^s}\right) & \text{otherwise}\ .\end{cases}$$

Proof of Theorem 1.6 . . For the elliptic curve $E_{D_T^2}\ :\ y^2=x^3-2^43^3D_T^2$, assume its period lattice is $L_T=\omega_TO_K$. Since the class number of $K=\mathbb{Q}(\sqrt{-3})$ is a $h_K=1,$ by [Sil 2] we know that all elliptic curves defined over $\mathbb{C}$ with complex multiplication ring O_K are $\mathbb{C}-$isomorphic each other. So their period lattices are Homothetic each other. We know the elliptic curve corresponding to the lattice O_K, denoted by $\mathbb{C}/O_K$, has complex multiplication ring O_K. Therefore every elliptic curve E over $\mathbb{C}$ with complex multiplication ring O_K has period lattice L Homothetic to O_K, i.e. we always have $L=\beta O_K$ (for some $\beta\in\mathbb{C}^\times$). Thus for the above elliptic curve $E_{D_T^2}$ and its period lattice $L_T=\omega_TO_K,$ we may assume $\omega_T=\alpha_T\omega,\ \alpha_T\in\mathbb{C}^\times$ (In fact, it's easy to see that $\omega_T=\omega(2\sqrt{3}\sqrt[3]{D_T})^{-1},$ i.e. $\alpha_T=1(2\sqrt{3}\sqrt[3]{D_T})^{-1}$). By [Ste, P.125] we know that the conductor of $\psi_{D_T^2}$ is $\sqrt{-3}D_T$ or $3D_T$. Therefore, in Proposition (A) above, putting $k=1,$

$\rho = \frac{\omega_T}{(3D)}$, $\mathfrak{h} = O_K$, $\mathfrak{g} = (3D)$, $\phi = \psi_{D_T^2}$, we obtain

$$\frac{\overline{\rho}}{|\rho|^{2s}} L_{\mathfrak{g}}(\overline{\psi}_{D_T^2}, s) = \sum_{\flat \in \mathbf{B}} H_1 \left(\frac{\psi_{D_T^2}(\flat)\omega_T}{3D}, 0, s, L_T \right), \quad (Re(s) > 3/2)$$

Since the conductor f of $\psi_{D_T^2}$ divides $(3D) = \mathfrak{g}$, so by Lemma B above we know the ray class of K to the cycle (modulo) $\mathfrak{g}$ is $K((E_{D_T^2})_{(3D)})$, In particular, we have

$$(O_K/(3D))^\times / \mu_6 \cong Gal\left(K((E_{D_T^2})_{(3D)})/K\right) \qquad (via \quad Artin \text{ map })$$

where $\mu_6 = \{\pm 1, \pm\tau, \pm\tau^2\}$ and $\mu_6 \cong (O_K/(3))^\times$. So we may take the set $\mathbf{B} = \{(3c + D) : \ c \in \mathcal{C}\}$, where $\mathcal{C}$ is a reduced residue system of O_K modulo D i.e. a representative system for $\mathcal{C}$is $(O_K/(D))^\times$. Thus

$$\frac{\overline{\rho}}{|\rho|^{2s}} L_{\mathfrak{g}}(\overline{\psi}_{D_T^2}, s) = \sum_{c \in \mathcal{C}} H_1 \left(\frac{\psi_{D_T^2}(3c + D)\omega_T}{3D}, 0, s, \omega_T O_K \right), \quad (Re(s) > 3/2)$$

Note that $H_1(z, 0, 1, L)$ could be analytically continued by the Eisenstein E^*-function (see [We 2]):

$$H_1(z, 0, 1, L) = E^*_{0,1}(z, L) = E^*_1(z, L).$$

Hence we have

$$\frac{1}{\rho} L_{\mathfrak{g}}(\overline{\psi}_{D_T^2}, 1) = \sum_{c \in \mathcal{C}} E^*_1 \left(\frac{\psi_{D_T^2}(3c + D)\omega_T}{3D}, \omega_T O_K \right), \quad (3.10)$$

that is

$$\frac{3D}{\alpha_T \ \omega} L_{\mathfrak{g}}(\overline{\psi}_{D_T^2}, 1) = \sum_{c \in \mathcal{C}} E^*_1 \left(\frac{\psi_{D_T^2}(3c + D)\alpha_T \ \omega}{3D}, \alpha_T \ \omega O_K \right).$$

Since $D \equiv 1(mod\ 6)$, so $3c + D \equiv 1(mod\ 3)$ for any $c \in \mathcal{C}$. Thus by the definition of $\psi_{D_T^2}$ and the cubic reciprocity, we have

$$\begin{aligned} \psi_{D_T^2}(3c + D) &= \overline{\left(\frac{D_T}{3c + D} \right)_3} (3c + D) = \overline{\left(\frac{3c + D}{D_T} \right)_3} (3c + D) \\ &= \overline{\left(\frac{3c}{D_T} \right)_3} (3c + D) = \left(\frac{3c}{D_T} \right)_3^2 (3c + D) \end{aligned}$$

Note that $L_{\mathfrak{g}}(\overline{\psi}_{D_T^2}, 1) = L_S(\overline{\psi}_{D_T^2}, 1)$, so by (3.10) we have

$$\frac{3D}{\alpha_T \ \omega} L_S(\overline{\psi}_{D_T^2}, 1) = \sum_{c \in \mathcal{C}} E^*_1 \left(\left(\frac{c \ \omega}{D} + \frac{\omega}{3} \right) \alpha_T \left(\frac{3c}{D_T} \right)_3^2, \quad \alpha_T \ \omega O_K \right) \ (3.11)$$

Let $\lambda = \alpha_T \left(\frac{3c}{D_T}\right)_3^2$, by formula $E_1^*(\lambda z, \lambda L) = \lambda^{-1} E_1^*(z, L)$, we obtain

$$\begin{aligned} E_1^*\left(\left(\frac{c\,\omega}{D} + \frac{\omega}{3}\right)\alpha_T \left(\frac{3c}{D_T}\right)_3^2 ,\ \alpha_T \omega O_K\right) &= E_1^*\left(\left(\frac{c\,\omega}{D} + \frac{\omega}{3}\right)\lambda\ ,\ \lambda L_\omega\right) \\ &= \lambda^{-1} E_1^*\left(\left(\frac{c\,\omega}{D} + \frac{\omega}{3}\right)\ ,\ L_\omega\right) \\ &= \alpha_T^{-1}\left(\frac{3c}{D_T}\right)_3 E_1^*\left(\left(\frac{c\,\omega}{D} + \frac{\omega}{3}\right)\ ,\ L_\omega\right). \end{aligned}$$

So by (3.11), we have

$$\begin{aligned} \frac{3D}{\omega} L_S(\overline{\psi}_{D_T^2}, 1) &= \sum_{c\ \in \mathcal{C}} \left(\frac{3c}{D_T}\right)_3 E_1^*\left(\left(\frac{c\,\omega}{D} + \frac{\omega}{3}\right)\ ,\ L_\omega\right) \\ &= \left(\frac{3}{D_T}\right)_3 \sum_{c\ \in \mathcal{C}} \left(\frac{c}{D_T}\right)_3 E_1^*\left(\left(\frac{c\,\omega}{D} + \frac{\omega}{3}\right)\ ,\ L_\omega\right), \end{aligned}$$

$$\frac{D}{\omega}\left(\frac{9}{D_T}\right)_3 L_S(\overline{\psi}_{D_T^2}, 1) = \frac{1}{3}\sum_{c\ \in \mathcal{C}} \left(\frac{c}{D_T}\right)_3 E_1^*\left(\left(\frac{c\,\omega}{D} + \frac{\omega}{3}\right)\ ,\ L_\omega\right), \quad (3.12)$$

By [Go-Sch, Prop.1.5] we know

$$E_1^*(z, L) = \zeta(z, L) - z s_2(L) - \overline{z} A(L)^{-1}$$

where

$$\zeta(z, L) = \frac{1}{z} + \sum_{\alpha \in L - \{0\}} \left(\frac{1}{z - \alpha} + \frac{1}{\alpha} + \frac{z}{\alpha^2}\right)$$

is Weierstrass $Zeta-$function, an odd function. For

$$S_2(L) = \lim_{s \to 0\ s > 0} \sum_{\alpha \in L - \{0\}} \alpha^{-2} |\alpha|^{-2s},$$

we have

$$\eta(\alpha, L) = \zeta(z + \alpha, L) - \zeta(z, L)\ ,\ \eta(\alpha, L) = \alpha S_2(L) + \overline{\alpha} A(L)^{-1}\ ,$$

(for any $\alpha \in L$), η is a quasi-period map corresponding to L. The Weierstrass $\wp-$function corresponding to the period lattice $L_\omega = \omega\ O_K$ is

$$\wp(z, L_\omega):\ \wp'(z)^2 = 4\wp(z)^3 - 1,$$

and in this case

$$A(L_\omega) = \frac{\overline{\omega}(\omega\tau) - \omega(\overline{\omega\tau})}{2\pi I} = \frac{\omega^2(\tau - \overline{\tau})}{2\pi I} = \frac{\sqrt{3}\omega^2}{2\pi}.$$

Also obviously we have $\frac{\omega}{2} \notin L_\omega$, therefore by [Sil 2 p.41] we know

$$\eta(\omega, L_\omega) = 2\zeta(\frac{\omega}{2}, L_\omega) \qquad \eta(\omega, L_\omega) = \omega S_2(L_\omega) + \overline{\omega} A(L_\omega)^{-1}.$$

Thus

$$2\zeta(\frac{\omega}{2}, L_\omega) = \omega S_2(L_\omega) + \overline{\omega} \cdot \frac{2\pi}{\sqrt{3}\omega^2} = \omega S_2(L_\omega) + \frac{2\pi}{\sqrt{3}\omega}.$$

So

$$S_2(L_\omega) = \frac{2}{\omega}\zeta(\frac{\omega}{2}, L_\omega) - \frac{2\pi}{\sqrt{3}\omega^2}.$$

Therefore

$$E_1^*(z, L_\omega) = \zeta(z, L_\omega) - \frac{2z}{\omega}\zeta(\frac{\omega}{2}, L_\omega) + \frac{2\pi z}{\sqrt{3}\omega^2} - \frac{2\pi}{\sqrt{3}\omega^2}\overline{z}.$$

Put $z = \frac{c\omega}{D} + \frac{\omega}{3}$, we obtain

$$\begin{aligned} E_1^*(\frac{c\omega}{D} + \frac{\omega}{3}, L_\omega) &= \zeta(\frac{c\omega}{D} + \frac{\omega}{3}, L_\omega) - \frac{2}{\omega}\left(\frac{c\omega}{D} + \frac{\omega}{3}\right)\zeta(\frac{\omega}{2}, L_\omega) \\ &\quad + \frac{2\pi}{\sqrt{3}\omega^2}\left(\left(\frac{c\omega}{D} + \frac{\omega}{3}\right) - \overline{\left(\frac{c}{D}\frac{\omega}{} + \frac{\omega}{3}\right)}\right) \\ &= \zeta(\frac{c\omega}{D} + \frac{\omega}{3}, L_\omega) - 2(\frac{c}{D} + \frac{1}{3})\zeta(\frac{\omega}{2}, L_\omega) + \frac{2\pi}{\sqrt{3}\omega}(\frac{c}{D} - \frac{\overline{c}}{\overline{D}}). \end{aligned}$$

That is

$$\begin{aligned} E_1^*(\frac{c\omega}{D} + \frac{\omega}{3}, L_\omega) &= \zeta(\frac{c\omega}{D} + \frac{\omega}{3}, L_\omega) \\ &\quad - 2(\frac{c}{D} + \frac{1}{3})\zeta(\frac{\omega}{2}, L_\omega) + \frac{2\pi}{\sqrt{3}\omega}(\frac{c}{D} - \frac{\overline{c}}{\overline{D}}) \end{aligned} \tag{3.13}$$

Now let us show $\zeta(\frac{\omega}{3}, L_\omega)$ and $\zeta(\frac{\omega}{2}, L_\omega)$ are equal. In fact, by formula (3.2) in [Ste, P.126], for any rational integer $D(3\dagger D)$, we know

$$\begin{aligned} L_D(s) &= \sum_\sigma \left(\frac{\sigma}{D}\right)_3 \sum_{u\in\mathbb{Q}(\tau)} \frac{3\Delta\overline{u} + 3\overline{\sigma} + \Delta}{N(3\Delta u + 3\sigma + \Delta)^s} \\ &= \frac{3\Delta}{N(3\Delta)^s}\left\{\sum_\sigma \left(\frac{\sigma}{D}\right)_3 \psi(\frac{\sigma}{\Delta} + \frac{1}{3}, s) + 6(1-s)\zeta_K(s)\sum_\sigma \left(\frac{\sigma}{D}\right)_3 \left(\frac{\overline{\sigma}}{\Delta} + \frac{1}{3}\right)\right\}. \end{aligned}$$

Let $s \to 1$, since $\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{\pi}{3\sqrt{3}}$, and $\psi(z, s, L)$ is convergent to $\xi(z, L)$ uniformly (when $s \to 1$), so

$$L_D(1) = \frac{1}{3\Delta}\left[\sum_\sigma \left(\frac{\sigma}{D}\right)_3 \xi(\frac{\sigma}{\Delta} + \frac{1}{3}) - \frac{2\pi}{\sqrt{3}}\sum_\sigma \left(\frac{\sigma}{D}\right)_3 (\frac{\overline{\sigma}}{\Delta} + \frac{1}{3})\right]$$

Put $u=\rho/3\Delta,\ \rho'=\rho+3\Delta,\quad \rho=3\sigma+\Delta$, then we have

$$\begin{aligned}
L_D(1) &= \frac{1}{3\Delta}\left[\sum_\sigma\left(\frac{\sigma}{D}\right)_3\xi(\frac{\rho}{3\Delta})-\frac{2\pi}{\sqrt{3}}\sum_\sigma\left(\frac{\sigma}{D}\right)_3\left(\frac{\overline{\rho}}{3\Delta}\right)\right]\\
&= \frac{1}{3\Delta}\left[\sum_{\sigma'}\left(\frac{\sigma'}{D}\right)_3\xi(\frac{\rho'}{3\Delta})-\frac{2\pi}{\sqrt{3}}\sum_{\sigma'}\left(\frac{\sigma'}{D}\right)_3\left(\frac{\overline{\rho'}}{3\Delta}\right)\right]\\
&= \frac{1}{3\Delta}\left[\sum_\sigma\left(\frac{\sigma}{D}\right)_3\xi(u+1)-\frac{2\pi}{\sqrt{3}}\sum_\sigma\left(\frac{\sigma}{D}\right)_3\left(\frac{\overline{\rho}}{3\Delta}+1\right)\right]\\
&= \frac{1}{3\Delta}\left[\sum_\sigma\left(\frac{\sigma}{D}\right)_3\xi(u+1)-\frac{2\pi}{\sqrt{3}}\sum_\sigma\left(\frac{\sigma}{D}\right)_3\left(\frac{\overline{\rho}}{3\Delta}\right)-\frac{2\pi}{\sqrt{3}}\sum_\sigma\left(\frac{\sigma}{D}\right)_3\right]
\end{aligned}$$

therefore we obtain

$$\frac{1}{3\Delta}(\xi(u+1)-\xi(u))=\frac{1}{3\Delta}\cdot\frac{2\pi}{\sqrt{3}}$$

that is $\xi(u+1)=\xi(u)+\frac{2\pi}{\sqrt{3}}$. Similarly we could obtain $\xi(u+\tau)=\xi(u)+\frac{2\pi}{\sqrt{3}}\overline{\tau}$. Therefore we have

$$\xi(u+1)=\xi(u)+\frac{2\pi}{\sqrt{3}}\ ,\qquad \xi(u+\tau)=\xi(u)+\frac{2\pi}{\sqrt{3}}\overline{\tau}\qquad (3.14)$$

where $\xi(z)$ is the Weierstrass $Zeta-$function with period lattice $L=\mathbb{Z}+\mathbb{Z}\tau=O_K$.

In formula (3.14), Let $u=-1/2$ or $-\frac{\tau}{2}$. Since $\xi(u)$ is an odd function , we obtain

$$\xi(\frac{1}{2})=\frac{\pi}{\sqrt{3}}\qquad \xi(\frac{\tau}{2})=\frac{\pi}{\sqrt{3}}\overline{\tau}\ ,$$

therefore

$$\zeta(\frac{\omega}{2},L_\omega)=\frac{1}{\omega}\xi(\frac{1}{2},O_K)=\frac{\pi}{\sqrt{3}\ \omega}\ ,$$

that is

$$\zeta(\frac{\omega}{2},L_\omega)=\frac{\pi}{\sqrt{3}\ \omega}\qquad (3.15)$$

Also by [Ste, P.127] we know,

$$\wp(\frac{\omega}{3},L_\omega)=1\ ,\quad \wp'(\frac{\omega}{3},\ L_\omega)=-\sqrt{3}\qquad (3.16)$$

so by the formulae

$$\wp''(z)=6\wp(z)^2-\frac{1}{2}g_2\qquad 2\zeta(2z)-4\zeta(z)=\frac{\wp''(z)}{\wp'(z)}\ ,$$

(see [Law], P.182), let $z=\omega/3$, we obtain $\wp''(\omega/3,L_\omega)=6\wp(\omega/3)^2=6$, so

$$2\zeta(2\omega/3,L_\omega)-4\zeta(\omega/3,L_\omega)=-6/\sqrt{3}=-2\sqrt{3},$$

i.e. $\zeta(2\omega/3, L_\omega) - 2\zeta(\omega/3, L_\omega) = -\sqrt{3}$. On the other hand, in formula (3.14), let $u = -1/3$, we obtain $\xi(-1/3+1) = \xi(-1/3) + 2\pi/\sqrt{3}$, i.e. $\xi(2/3) + \xi(1/3) = 2\pi/\sqrt{3}$. Also we have

$$\zeta(2\omega/3, L_\omega) = \omega - 1\xi(2/3), \qquad \zeta(\omega/3, L_\omega) = \omega^{-1}\xi(1/3),$$

so

$$\omega\zeta(\frac{2\omega}{3}, L_\omega) + \omega\zeta(\frac{\omega}{3}, L_\omega) = \frac{2\pi}{\sqrt{3}} ,$$

which gives

$$\begin{cases} \zeta(2\omega/3, L_\omega) + \zeta(\omega/3, L_\omega) = 2\pi/(\sqrt{3}\omega), \\ \zeta(2\omega/3, L_\omega) - 2\zeta(\omega/3, L_\omega) = -\sqrt{3}. \end{cases}$$

This gives the solution

$$\zeta(\frac{\omega}{3}, L_\omega) = \frac{2\pi}{3\sqrt{3}\omega} + \frac{1}{\sqrt{3}} , \quad \zeta(\frac{2\omega}{3}, L_\omega) = \frac{4\pi}{3\sqrt{3}\omega} - \frac{1}{\sqrt{3}} \quad . \quad (3.17)$$

Also by the formula (see [Law]):

$$\zeta(z_1 + z_2, L_\omega) = \zeta(z_1, L_\omega) + \zeta(z_2, L_\omega) + \frac{1}{2}\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} ,$$

we obtain

$$\begin{aligned} \zeta(\frac{c\ \omega}{D} + \frac{\omega}{3}, L_\omega) &= \zeta(\frac{c\ \omega}{D}, L_\omega) + \zeta(\frac{\omega}{3}, L_\omega) + \frac{1}{2}\frac{\wp'(\frac{c\ \omega}{D}, L_\omega) - \wp'(\frac{\omega}{3}, L_\omega)}{\wp(\frac{c\ \omega}{D}, L_\omega) - \wp(\frac{\omega}{3}, L_\omega)} \\ &= \zeta(\frac{c\ \omega}{D}, L_\omega) + \frac{2\pi}{3\sqrt{3}\omega} + \frac{1}{\sqrt{3}} + \frac{1}{2}\frac{\wp'(\frac{c\ \omega}{D}, L_\omega) + \sqrt{3}}{\wp(\frac{c\ \omega}{D}, L_\omega) - 1} \end{aligned}$$

substitute this into (3.13), we obtain

$$\begin{aligned} E_1^*(\frac{c\omega}{D} + \frac{\omega}{3}, L_\omega) &= \zeta(\frac{c\ \omega}{D}, L_\omega) + \frac{2\pi}{3\sqrt{3}\omega} + \frac{1}{\sqrt{3}} + \frac{1}{2}\frac{\wp'(\frac{c\ \omega}{D}, L_\omega) + \sqrt{3}}{\wp(\frac{c\ \omega}{D}, L_\omega) - 1} \\ &\quad - 2(\frac{c}{D} + \frac{1}{3})\frac{\pi}{\sqrt{3}\omega} + \frac{2\pi}{\sqrt{3}\omega}(\frac{c}{D} - \frac{\overline{c}}{\overline{D}}) \\ &= \zeta(\frac{c\ \omega}{D}, L_\omega) + \frac{1}{2}\frac{\wp'(\frac{c\ \omega}{D}, L_\omega) + \sqrt{3}}{\wp(\frac{c\ \omega}{D}, L_\omega) - 1} + \frac{1}{\sqrt{3}} - \frac{2\pi}{\sqrt{3}\ \omega} \cdot \frac{\overline{c}}{\overline{D}} \end{aligned}$$

Now substitute this into (3.12), we have

$$\begin{aligned} &\frac{D}{\omega}\left(\frac{9}{D_T}\right)_3 \ L_S(\overline{\psi}_{D_T^2}, 1) \\ &= \frac{1}{3}\sum_{c\ \in\mathcal{C}}\left(\frac{c}{D_T}\right)_3 \left[\zeta(\frac{c\ \omega}{D}, L_\omega) + \frac{1}{2}\frac{\wp'(\frac{c\ \omega}{D}, L_\omega) + \sqrt{3}}{\wp(\frac{c\ \omega}{D}, L_\omega) - 1} + \frac{1}{\sqrt{3}} - \frac{2\pi}{\sqrt{3}\ \omega} \cdot \frac{\overline{c}}{\overline{D}}\right]. \end{aligned}$$

Since $D = \pi_1 \cdots \pi_n$ with $\pi_k \equiv 1 (mod\ 6)$, so we may choose the representatives $\mathcal{C}$ for $(O_K/(D))^\times$ in such a way that $-c \in \mathcal{C}$ when $c \in \mathcal{C}$. Obviously $(-c/D_T)_3 = (c/D_T)_3$. Also since $\zeta(z, L_\omega)$ and $\wp'(z, L_\omega)$ are odd functions , and $\wp(z, L_\omega)$ is even function, so

$$\sum_{c\ \in \mathcal{C}} \left(\frac{c}{D_T}\right)_3 \zeta(\frac{c\ \omega}{D}, L_\omega) = \sum_{c\ \in \mathcal{C}} \left(\frac{c}{D_T}\right)_3 \frac{\wp'(\frac{c\ \omega}{D}, L_\omega)}{\wp(\frac{c\ \omega}{D}, L_\omega) - 1}$$
$$= \sum_{c\ \in \mathcal{C}} \left(\frac{c}{D_T}\right)_3 \cdot \frac{\overline{c}}{\overline{D}} = 0$$

Therefore ,

$$\frac{D}{\omega}\left(\frac{9}{D_T}\right)_3 L_S(\overline{\psi}_{D_T^2}, 1) = \frac{1}{2\sqrt{3}} \sum_{c\ \in \mathcal{C}} \left(\frac{c}{D_T}\right)_3 \frac{1}{\wp\left(\frac{c\omega}{D}\right) - 1} + \frac{1}{3\sqrt{3}} \sum_{c\ \in \mathcal{C}} \left(\frac{c}{D_T}\right)_3 .$$

This proves Theorem 1.6.

Lemma 3.2.

$$\sum_{c\ \in \mathcal{C}} \left(\frac{c}{D_T}\right)_3 = \begin{cases} \sharp\mathcal{C} & \text{if} \quad T = \emptyset; \\ 0 & \text{if} \quad T \neq \emptyset. \end{cases}$$

Proof . This could be verified by the definition of cubic residue symbol(see [Ire-Ro]).

Lemma 3.3. $\sum\limits_T \left(\frac{c}{D_T}\right)_3 = \mu \cdot 2^t,$

where $\mu \in \{\pm 1, \pm\tau \pm \tau^2\}$, $t = \sharp\{\pi :\ \pi \equiv 1 (mod\ 6)$ is a prime element, $\pi | D$, and $(c/\pi)_3 = 1\}$, $c \in O_K$ and D are relatively prime.

Proof . By $\sum\limits_T \left(\frac{c}{D_T}\right)_3 = \left(1 + \left(\frac{c}{\pi_1}\right)_3\right) \cdots \left(1 + \left(\frac{c}{\pi_n}\right)_3\right)$ and the definition of cubic residue symbol, the lemma could be verified easily.

Lemma 3.4.

$$(1) \sum_T 2^{n-t(T)} \left(\frac{c}{D_T}\right)_3 = (-\tau^2)^{t_\tau} \cdot 3^{t_1} (1-\tau)^{t_\tau + t_{\tau^2}} \quad ;$$

$$(2) \quad \sum_T (-1)^{t(T)} \left(\frac{c}{D_T}\right)_3 = \begin{cases} 0 & \text{if} \quad t_\tau + t_{\tau^2} < n \quad ; \\ (1-\tau)^{t_\tau + t_{\tau^2}} \cdot (-\tau^2)^{t_{\tau^2}} & \text{if} \quad t_\tau + t_{\tau^2} = n \end{cases} .$$

$$(3) \quad \sum_T \sum_{c \in \mathcal{C}} 2^{n-t(T)} \left(\frac{c}{D_T}\right)_3 = 2^n \cdot \sharp\mathcal{C}.$$

where $c \in O_K$, and D are relatively prime, the sum $\sum\limits_T$ is taken for T runs over subsets of $\{1, \cdots, n\}$, $t = t(T) = \sharp T$(but $t = 0$ when $T = \emptyset$),

$$t_1 = \sharp\left\{\pi_k :\ \left(\frac{c}{\pi_k}\right)_3 = 1\right\}, \quad t_\tau = \sharp\left\{\pi_k :\ \left(\frac{c}{\pi_k}\right)_3 = \tau\right\}, \quad t_{\tau^2} = \sharp\left\{\pi_k :\ \left(\frac{c}{\pi_k}\right)_3 = \tau^2\right\}$$

$t_1 + t_\tau + t_{\tau^2} = n$.

Proof . Note that

$$\sum_T 2^{n-t(T)} \left(\frac{c}{D_T}\right)_3 = \left(2 + \left(\frac{c}{\pi_1}\right)_3\right) \cdots \left(2 + \left(\frac{c}{\pi_n}\right)_3\right)$$

$$\sum_T (-1)^{n-t(T)} \left(\frac{c}{D_T}\right)_3 = \left(1 - \left(\frac{c}{\pi_1}\right)_3\right) \cdots \left(1 - \left(\frac{c}{\pi_n}\right)_3\right).$$

Then by the definition of cubic residue symbol and Lemma 3.2, we could obtain the results.

Lemma 3.5 . For the Weierstrass $\wp$−function $\wp(z, L_\omega)$ in Theorem 4.3.1 and any $c \in \mathcal{C}$, we have

$$v_3\left(\wp\left(\frac{c\,\omega}{D},\ L_\omega\right) - 1\right) = \frac{1}{3}.$$

Proof . We need the following two lemmas from [Ste , P.128]:

Lemma (C). Let Δ be a square-free integer in $K = \mathbb{Q}(\sqrt{-3})$ relatively prime to $\sqrt{-3}$, $\beta \in O_K$ is relatively prime to Δ . And

$$\varphi(\Delta) = \begin{cases} \Delta^{2/(\Delta\overline{\Delta}-1)} & \text{if } \Delta \text{ is a prime ;} \\ 1 \quad , & \text{otherwise .} \end{cases}$$

Then $\varphi(\Delta)\wp(\beta\omega/\Delta)$ is a unit.

Lemma (D). Let $r > 0$ be a rational integer. Assume β, Δ, and γ are integers of $K = \mathbb{Q}(\sqrt{-3})$, both Δ and γ are relatively prime to $\sqrt{-3}$, andβ is relatively prime to Δ. Let $\lambda = \frac{1}{2}(1 - 3^{1-r})$, $\varphi(\Delta)$ be as in Lemma (C). Then

$$3^{-\lambda}\varphi(\Delta)\left\{\wp\left(\beta\omega/\Delta\right) - \wp\left(\gamma\omega/(\sqrt{-3})^r\right)\right\}$$

is a unit.

Now we turn to the proof of Lemma 3.5. In Lemma (D), let $r = 2$, $\gamma = -1$, $\Delta = D$, $\beta = c$, then $\lambda = \frac{1}{2}(1 - 3^{1-2}) = \frac{1}{3}$, and we know

$$3^{-\frac{1}{3}}\varphi(D)\left\{\wp\left(\frac{c\,\omega}{D}\right) - \wp\left(-\omega/(\sqrt{-3})^2\right)\right\} = \theta$$

is a unit. $\varphi(D) = \varphi(\Delta)$ is as in(C). Since $\wp\left(-\omega/(\sqrt{-3})^2\right) = \wp(\omega/3) = 1$, so

$$\wp\left(\frac{c\,\omega}{D}\right) - 1 = \wp\left(\frac{c\,\omega}{D}\right) - \wp\left(-\omega/(\sqrt{-3})^2\right) = 3^{\frac{1}{3}}\varphi(D)^{-1}\theta$$

Thus we have $v_3(\varphi(D)) = 0$, and

$$v_3\left(\wp\left(\frac{c\,\omega}{D}\right) - 1\right) = v_3(3^{\frac{1}{3}}\varphi(D)^{-1}\theta) = \frac{1}{3}.$$

This proves Lemma 3.5 .

Proof of Theorem 1.7 . For each subset T of $\{1, \cdots, n\}$, multiply the two sides of formula (1.6) in Theorem 1.6 by $2^{n-t(T)}$, and then add them up, we obtain

$$\sum_T 2^{n-t(T)} \frac{D}{\omega} \left(\frac{9}{D_T}\right)_3 L_S(\overline{\psi}_{D_T^2}, 1)$$
$$= \frac{1}{2\sqrt{3}} \sum_T 2^{n-t(T)} \sum_{c\ \in \mathcal{C}} \left(\frac{c}{D_T}\right)_3 \frac{1}{\wp\left(\frac{c\ \omega}{D}\right)-1} + \frac{1}{3\sqrt{3}} \sum_T 2^{n-t(T)} \sum_{c\ \in \mathcal{C}} \left(\frac{c}{D_T}\right)_3 .$$

Then by Lemma 3.4(3), we have

$$\sum_T 2^{n-t(T)} \frac{D}{\omega} \left(\frac{9}{D_T}\right)_3 L_S(\overline{\psi}_{D_T^2}, 1)$$
$$= \frac{1}{2\sqrt{3}} \sum_{c\ \in \mathcal{C}} \frac{1}{\wp\left(\frac{c\ \omega}{D}\right)-1} \sum_T 2^{n-t(T)} \left(\frac{c}{D_T}\right)_3 + \frac{2^n}{3\sqrt{3}} \cdot \sharp\ \mathcal{C} \qquad (3.18)$$

By Lemma 3.4(1) we know,

$$\sum_T 2^{n-t(T)} \left(\frac{c}{D_T}\right)_3 = (-\tau^2)^{t_\tau} \cdot 3^{t_1} (1-\tau)^{t_\tau + t_{\tau^2}} ,$$

Note that $1-\tau$ and $\sqrt{-3}$ are associated each other (Both are prime elements in O_K), so $v_3(1-\tau) = v_3(\sqrt{-3}) = \frac{1}{2}$. Hence

$$v_3\left(\sum_T 2^{n-t(T)} \left(\frac{c}{D_T}\right)_3 \right) = t_1 + \frac{t_\tau + t_{\tau^2}}{2} = \frac{n+t_1}{2} \geq \frac{n}{2}$$

Also we have $\sharp\ \mathcal{C} = \prod_{k=1}^{n} (\pi_k \overline{\pi_k} - 1)$, so by the assumption we have $v_3(\sharp\ \mathcal{C}) \geq n$, therefore

$$v_3\left(\frac{2^n}{3\sqrt{3}} \cdot \sharp\ \mathcal{C}\right) \geq n - \frac{3}{2} \geq \frac{n}{2} - 1 \qquad (n \geq 1)$$

so by Lemma 3.5 we know that the first term in the right side of (3.18) has 3−adic valuation $\geq \frac{n}{2} - \frac{1}{3} - \frac{1}{2} = \frac{n}{2} - \frac{5}{6} > \frac{n}{2} - 1$. Therefore the right side of (3.18) has 3−adic valuation $\geq \frac{n}{2} - 1$. Hence

$$v_3\left(\sum_T 2^{n-t(T)} \frac{D}{\omega} \left(\frac{9}{D_T}\right)_3 L_S(\overline{\psi}_{D_T^2}, 1)\right) \geq \frac{n}{2} - 1.$$

Also by Lemma 3.1 we know that if $T = \{1, \cdots, n\}$ then $L_S(\overline{\psi}_{D_T^2}, 1) = L(\overline{\psi}_{D^2}, 1)$; and if $T = \emptyset$ then $L_S(\overline{\psi}_{D_T^2}, 1) = L_S(\overline{\psi}_1, 1) = L(\overline{\psi}_1, 1) \prod_{k=1}^{n} \left(1 - \frac{1}{\pi_k}\right)$. From [Ste] we know

$$L(\overline{\psi}_1, 1) = L(\psi_1,\ 1) = L_1(1) = (\frac{\sqrt{3}}{9})\omega ,$$

where ψ_1 is the Hecke characters of the number field $K=\mathbb{Q}(\sqrt{-3})$ corresponding to the elliptic curve $E:\ y^2=x^3-2^4 3^3$. Thus $L_S(\overline{\psi}_1,1)=\frac{\sqrt{3}}{9}\omega\prod_{k=1}^{n}\left(1-\frac{1}{\pi_k}\right)$. So we have

$$\begin{aligned} v_3\left(L_S(\overline{\psi}_1,\ 1)/\omega\right) &= v\left(\frac{\sqrt{3}}{9}\right)+\sum_{k=1}^{n} v_3(\pi_k-1) \\ &\geq n-\frac{3}{2}\geq \frac{n}{2}-1 \quad (\text{ Since } v_3\ (\pi_k-1)\geq 1) \end{aligned}$$

Now we use induction method on n to prove $v_3\left(L(\overline{\psi}_{D^2},\ 1)/\omega\right)\geq \frac{n}{2}-1$. when $n=1$, $D=\pi_1$, and$L_S(\overline{\psi}_1,\ 1)=\frac{\sqrt{3}}{9}\omega\left(1-\frac{1}{\pi_1}\right)$. Since $v_3(\pi_1-1)\geq 1$, so

$$v_3\left(L_S(\overline{\psi}_1,\ 1)/\omega\right)\geq 1-\frac{3}{2}=-\frac{1}{2}.$$

Also we have

$$v_3\left(2\frac{\pi_1}{\omega}\left(\frac{9}{D_\emptyset}\right)_3 L_S(\overline{\psi}_1,\ 1)+\frac{\pi_1}{\omega}\left(\frac{9}{\pi_1}\right)_3 L(\overline{\psi}_{\pi_1^2},\ 1)\right)\geq \frac{1}{2}-1=-\frac{1}{2}$$

Therefore

$$v_3\left(L(\overline{\psi}_{\pi_1^2},\ 1)/\omega\right)=v_3\left(\frac{\pi_1}{\omega}\left(\frac{9}{\pi_1}\right)_3 L(\overline{\psi}_{\pi_1^2},\ 1)\right)\geq -\frac{1}{2}=\frac{1}{2}-1.$$

Assume our conclusion is true for $1,\ 2,\ \cdots,\ n-1$, and consider the case n, $D=\pi_1\cdots\pi_n$. For any nonempty subset T of $\{1,\ \cdots,n\}$, put $t=t(T)=\sharp\ T$, by Lemma 3.1 we know

$$2^{n-t(T)}\ \frac{D}{\omega}\left(\frac{9}{D_T}\right)_3 L_S(\overline{\psi}_{D_T^2},1)=2^{n-t(T)}\ \frac{D}{\omega}\left(\frac{9}{D_T}\right)_3 L(\overline{\psi}_{D_T^2},1)\prod_{\pi_k|\widehat{D}_T}\left(1-\left(\frac{D_T}{\pi_k}\right)_3\frac{1}{\pi_k}\right)$$

Hence

$$\begin{aligned} v_3\left(2^{n-t(T)}\ \frac{D}{\omega}\left(\frac{9}{D_T}\right)_3 L_S(\overline{\psi}_{D_T^2},1)\right) &= v_3\left(L_S(\overline{\psi}_{D_T^2},1)/\omega\right) \\ &= v_3\left(L(\overline{\psi}_{D_T^2},1)/\omega\right)+\sum_{\pi_k|\widehat{D}_T} v_3\left(1-\left(\frac{D_T}{\pi_k}\right)_3\frac{1}{\pi_k}\right) \end{aligned}$$

Note that T is a proper subset, by induction assumption we have $v_3\left(L(\overline{\psi}_{D_T^2},1)/\omega\right)\geq \frac{t(T)}{2}-1$. Since $\left(\frac{D_T}{\pi_k}\right)_3=1,\ \tau,$ or τ^2 (since $\pi_k|\widehat{D}_T$), so

$$v_3\left(1-\left(\frac{D_T}{\pi_k}\right)_3\frac{1}{\pi_k}\right)=v_3(\pi_k-1)\ ,\ v_3(\pi_k-\tau)\ ,\ \text{or}\ \ v_3(\pi_k-\tau^2)\geq 1\ ,\ \frac{1}{2}\ ,\ \text{or}\ \frac{1}{2}\quad .$$

thus

$$\sum_{\pi_k|\widehat{D}_T} v_3\left(1-\left(\frac{D_T}{\pi_k}\right)_3\frac{1}{\pi_k}\right)\geq \frac{n-t(T)}{2}.$$

Therefore
$$v_3\left(2^{n-t(T)}\ \frac{D}{\omega}\left(\frac{9}{D_T}\right)_3 L_S(\overline{\psi}_{D_T^2},1)\right)\geq \frac{t(T)}{2}-1+\frac{n-t(T)}{2}=\frac{n}{2}-1.$$
Also when $T=\emptyset$, we above have proved
$$v_3\left(2^{n}\ \frac{D}{\omega}\left(\frac{9}{D_\emptyset}\right)_3 L_S(\overline{\psi}_1,1)\right)=v_3\left(L_S(\overline{\psi}_1,\ 1)/\omega\right)\geq\frac{n}{2}-1.$$

Therefore

$$\begin{aligned}
&v_3\left(L(\overline{\psi}_{D^2},\ 1)/\omega\right)\\
&\qquad=v_3\left(2^{n-n}\ \frac{D}{\omega}\left(\frac{9}{D}\right)_3 L_S(\overline{\psi}_{D^2},1)\right)\\
&\qquad=v_3\left(\left(\sum_T 2^{n-t(T)}\ \frac{D}{\omega}\left(\frac{9}{D_T}\right)_3 L_S(\overline{\psi}_{D_T^2},1)\right)\right.\\
&\qquad\qquad\left.-\left(\sum_{T\subsetneqq\{1,\cdots,n\}} 2^{n-t(T)}\ \frac{D}{\omega}\left(\frac{9}{D_T}\right)_3 L_S(\overline{\psi}_{D_T^2},1)\right)\right)\\
&\qquad\geq\frac{n}{2}-1
\end{aligned}$$

This proves our conclusion by induction, and completes the proof of Theorem 1.7 .

Proof of Theorem 1.8 . Since $\pi_k\equiv 1(mod\ 6),\ \sharp\,\mathcal{C}=\prod_{k=1}^{n}(\pi_k\overline{\pi}_k-1)\equiv 0(mod\ 6),$ so we may choose the set $\mathcal{C},$ in such a way that $\pm c,\ \pm\tau c,\ \pm\tau^2 c\in\mathcal{C}($ when $c\in\mathcal{C})$. That is, when $c\in\mathcal{C}$, then all its associated elements are in $\mathcal{C}$. Let $V=\{c\in\mathcal{C}:\ c\equiv 1(mod\ 3)\},$ Then
$$\mathcal{C}=\bigcup_{\mu\ \in\ \{\pm1,\pm\tau,\pm\tau^2\}}\mu V\quad .$$

Since
$$\left(\frac{-c}{D_T}\right)_3=\left(\frac{c}{D_T}\right)_3\ ,\qquad\left(\frac{\tau}{D_T}\right)_3=1,\ \tau,\ \text{or}\ \ \tau^2\quad ,$$
$$\wp\left(\frac{-c\ \omega}{D},\ L_\omega\right)=\wp\left(\frac{c\ \omega}{D},\ L_\omega\right)\quad ,$$
$$\wp(\tau z,\ L_\omega)=\wp(\tau z,\ \tau L_\omega)=\frac{1}{\tau^2}\wp(z,\ L_\omega)=\tau\wp(z,\ L_\omega)\quad ,$$
$$\wp(\tau^2 z,\ L_\omega)=\wp(\tau^2 z,\ \tau^2 L_\omega)=\frac{1}{\tau^4}\wp(z,\ L_\omega)=\tau^2\wp(z,\ L_\omega)\quad ,$$

therefore

$$
\begin{aligned}
S^*(D) &= \frac{1}{\sqrt{3}} \sum_{c\ \in V} \frac{1}{\wp(\frac{c\ \omega}{D},\ L_\omega)-1} \sum_T 2^{n-t(T)} \ \left(\frac{c}{D_T}\right)_3 \\
&+ \frac{1}{\sqrt{3}} \sum_{c\ \in V} \frac{1}{\wp(\frac{\tau\ c\ \omega}{D},\ L_\omega)-1} \sum_T 2^{n-t(T)} \ \left(\frac{\tau\ c}{D_T}\right)_3 \\
&+ \frac{1}{\sqrt{3}} \sum_{c\ \in V} \frac{1}{\wp(\frac{\tau^2\ c\ \omega}{D},\ L_\omega)-1} \sum_T 2^{n-t(T)} \ \left(\frac{\tau^2\ c}{D_T}\right)_3 \\
&= \frac{1}{\sqrt{3}} \sum_T 2^{n-t(T)} \left\{ \sum_{c\ \in V} W_c \left(\frac{c}{D_T}\right)_3 \right\}
\end{aligned}
$$

where

$$
W_c = \frac{1}{\wp(\frac{c\ \omega}{D},\ L_\omega)-1} + \frac{\left(\frac{\tau}{D_T}\right)_3}{\tau\ \wp(\frac{c\ \omega}{D},\ L_\omega)-1} + \frac{\left(\frac{\tau}{D_T}\right)_3^2}{\tau^2\ \wp(\frac{c\ \omega}{D},\ L_\omega)-1}.
$$

Denote

$$
U(\wp) = \left(\wp\left(\frac{c\ \omega}{D}\right)-1\right)\left(\tau\ \wp\left(\frac{c\ \omega}{D}\right)-1\right)\left(\tau^2\ \wp\left(\frac{c\ \omega}{D}\right)-1\right)
$$

then

$$
\begin{aligned}
&\frac{1}{\wp(\frac{c\ \omega}{D},\ L_\omega)-1} + \frac{\left(\frac{\tau}{D_T}\right)_3}{\tau\ \wp(\frac{c\ \omega}{D},\ L_\omega)-1} + \frac{\left(\frac{\tau}{D_T}\right)_3^2}{\tau^2\ \wp(\frac{c\ \omega}{D},\ L_\omega)-1} \\
&= \frac{1}{U(\wp)} \cdot \begin{cases} 3 & \text{if} \quad \left(\frac{\tau}{D_T}\right)_3 = 1 \quad ; \\ 3\wp^2 & \text{if} \quad \left(\frac{\tau}{D_T}\right)_3 = \tau \quad ; \\ 3\wp & \text{if} \quad \left(\frac{\tau}{D_T}\right)_3 = \tau^2 \quad . \end{cases}
\end{aligned}
$$

Also put

$$
V(\wp) = \begin{cases} 1 & \text{if} \quad \left(\frac{\tau}{D_T}\right)_3 = 1 \quad ; \\ \wp^2 & \text{if} \quad \left(\frac{\tau}{D_T}\right)_3 = \tau \quad ; \\ \wp & \text{if} \quad \left(\frac{\tau}{D_T}\right)_3 = \tau^2 \quad . \end{cases}
$$

where $\quad \wp = \wp\left(\frac{c\ \omega}{D}\right)$, then

$$
\begin{aligned}
S^*(D) &= \frac{1}{\sqrt{3}} \sum_T 2^{n-t(T)} \left\{ \sum_{c\ \in V} \frac{3V(\wp)}{U(\wp)} \cdot \left(\frac{c}{D_T}\right)_3 \right\} \\
&= \sqrt{3} \sum_{c\ \in V} \frac{V(\wp)}{U(\wp)} \sum_T 2^{n-t(T)} \left(\frac{c}{D_T}\right)_3
\end{aligned}
$$

Since

$$
\begin{aligned}
\tau\ \wp\left(\frac{c\ \omega}{D}\right)-1 &= \left(\wp\left(\frac{c\ \omega}{D}\right)-\tau^2\right)/\tau^2 = \frac{1}{\tau^2}\left(\left(\wp\left(\frac{c\ \omega}{D}\right)-1\right)+(1-\tau^2)\right) \ , \\
\tau^2\ \wp\left(\frac{c\ \omega}{D}\right)-1 &= \frac{1}{\tau}\left(\wp\left(\frac{c\ \omega}{D}\right)-\tau\right) = \frac{1}{\tau}\left(\left(\wp\left(\frac{c\ \omega}{D}\right)-1\right)+(1-\tau)\right) \ ,
\end{aligned}
$$

so by Lemma 3.5 we obtain

$$v_3\left(\tau\ \wp\left(\frac{c\ \omega}{D}\right)-1\right)=v_3\left(\tau^2\ \wp\left(\frac{c\ \omega}{D}\right)-1\right)=v_3\left(\wp\left(\frac{c\ \omega}{D}\right)-1\right)=\frac{1}{3}\ ,$$

thus

$$v_3(U(\wp))=3v_3\left(\wp\left(\frac{c\ \omega}{D}\right)-1\right)=3\cdot\frac{1}{3}=1\quad .$$

Also we have

$$v_3\left(\sum_T 2^{n-t(T)}\left(\frac{c}{D_T}\right)_3\right)\geq\frac{n}{2}\quad (\text{ see Lemma 3.4(1)})$$

therefore

$$v_3(S^*(D))\geq\frac{1}{2}+v_3\left(\frac{V(\wp)}{U(\wp)}\right)+v_3\left(\sum_T 2^{n-t(T)}\left(\frac{c}{D_T}\right)_3\right)\geq\frac{1}{2}-1+\frac{n}{2}=\frac{n-1}{2}\quad .$$

This proves the Proposition .

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(This paper was published in :
Acta Arithmetica, 103.1(2002), 79-95);
Manuscripta Math. 108, 385-397 (2002)